Answers:

1. D
2. D
3. C
4. B
5. A
6. B
7. B
8. E
9. D
10. E
11. D
12. A
13. C
14. A
15. C
16. D
17. B
18. E
19. D
20. A
21. B
22. C
23. A
24. B
25. E
26. B
27. A
28. A
29. A
30. D

Solutions:

1. $221063 = 43 \cdot 53 \cdot 97$ and $218929 = 37 \cdot 61 \cdot 97$, so the greatest common divisor of the two numbers is 97. Since $102 = 2 \cdot 3 \cdot 17$, its number of positive integral divisors is $2 \cdot 2 \cdot 2 = 8$.

2. The smallest two perfect numbers are 6 and 28, so the number of positive integral divisors of $6 + 28 = 34 = 2 \cdot 17$ is $2 \cdot 2 = 4$.

3. This is the set of integers, commonly denoted $\mathbb{Z}$. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{Q}$ are the symbols for the natural numbers, real numbers, and rational numbers, respectively.

4. Looking at the hundreds' place, either $3 + A = C$ or $4 + A = C$ if carrying was necessary from the tens' place. Assuming the latter, either $A + C = 17$ or $A + C = 16$ if carrying was necessary from the ones' place. However, solving $4 + A = C$ with either of these yields non-integer solutions (first equation) or $C = 10$ (second equation), both of which are not possible. Therefore, $3 + A = C$ is true, making $A + C = 7$ or $A + C = 6$ if carrying was necessary from the ones' place. However, solving $3 + A = C$ with the second of these equations yields non-integer solutions. Therefore, $3 + A = C$ and $A + C = 7$, and solving these two equation yields $A = 2$ and $C = 5$. Since no carrying was necessary from the ones' place, $4 + B = C = 5 \Rightarrow B = 1$. Thus, $A + B + C = 2 + 1 + 5 = 8$ (verifying the original problem, $324 + 1251 = 1575$).

5. Solving the first and last equivalences yields $x \equiv 4 (\text{mod} 33)$, and solving the second and third equivalences yields $x \equiv 17 (\text{mod} 35)$. Solving these two equivalences yields $x \equiv 367 (\text{mod} 1155)$, so the two smallest positive integral solutions are 367 and $367 + 1155 = 1522$, and the sum of those solutions is $367 + 1522 = 1889$.

6. The triangular numbers are numbers of the form $\frac{n(n+1)}{2}$, and the square numbers are numbers of the form n^2, where n is a positive integer. Checking the first several numbers of both forms, the first positive integer that appears in both lists is 36.

7. First, convert 35:56:234 into a standard time: $35{:}56{:}234 = 35{:}59{:}54$. Now, adding the times together, $7{:}34{:}52 + 35{:}59{:}54 = 42{:}93{:}106 = 42{:}94{:}46 = 43{:}34{:}46$, which would show the same time as if the sum had been $19{:}34{:}46$.

8. The 7th element would be $\binom{15}{6} = 5005$.

9. Since $4096=2^{12}$, the sum is $1-2+4-8+16-32+64-128+256-512+1024$ $-2048+4096=2731$.

10. The greatest number of positive integral divisors a number less than 100 has is 12, which occurs for 60, 72, 84, 90, and 96. $60+72+84+90+96=402$

11. $x_{10}=221_a-101_b=\left(2a^2+2a+1\right)-\left(b^2+1\right)=2a^2+2a-b^2=2a^2+2a-\left(a+8\right)^2$ $=2a^2+2a-a^2-16a-64=a^2-14a-64=\left(a-7\right)^2-113$, so the minimum value of x is -113, which occurs when $a=7$ and $b=15$.

12. $x\equiv-1\left(\text{mod}2\right)$ and $x\equiv-1\left(\text{mod}3\right)$, so because 2 and 3 are relatively prime, $x\equiv-1\left(\text{mod}6\right)$. Therefore, the solutions that fit the interval are 35, 41, 47, 53, 59, 65, and 71, making 7 total.

13. According to the definition, the Fibonacci sequence, with negative subscripted terms included, would look like $\ldots,-21,13,-8,5,-3,2,-1,1,0,1,1,2,3,5,8,13,21,\ldots$, and it is easy to see that $F_{-8}+F_8=-21+21=0$.

14. $\sum_{n=1}^{10}a_n=1+2+2+3+2+4+2+4+3+4=27$

15. Since $260-6\cdot43=2$, we need only find the solution to the equivalence $2\cdot x\equiv1\left(\text{mod}43\right)\Rightarrow2\cdot x\equiv44\left(\text{mod}43\right)\Rightarrow x\equiv22\left(\text{mod}43\right)$. Thus, the smallest positive integer solution to the equivalence is 22, and the sum of its digits is $2+2=4$.

16. In modulus 100, $2^{96}\equiv512^{10}2^6\equiv12^{10}\left(64\right)\equiv144^5\left(64\right)\equiv44^5\left(64\right)\equiv1936^2\left(44\right)\left(64\right)$ $\equiv36^2\left(2816\right)\equiv1296\left(16\right)\equiv96\left(16\right)\equiv1536\equiv36$, so the last two digits are 36.

17. Since $f\left(x\right)$ is divisible by $x+1$, $0=f\left(-1\right)=-1+b-2-c\Rightarrow b-c=3$. Since $f\left(x\right)$ leaves a remainder of 12 when divided by $x-2$, $12=f\left(2\right)=32+4b+4-c$ $\Rightarrow4b-c=-24$. Solving this system yields $b=-9$ and $c=-12$. The remainder when $f\left(x\right)$ is divided by $x+2$ is $f\left(-2\right)=-32-36-4+12=-60$.

18. First, write $14w+12x+24y+26z=14w+12x+12\left(2y\right)+14z+12z=14\left(w+z\right)$

$+12(x+2y+z)=14w'+12x'$, where $w'=w+z$ and $x'=x+2y+z$, so basically we can just consider the form as a linear combination of $14w+12x=2(7w+6x)$. Now, this is like the Frobenius problem, and because 7 and 6 are relatively prime, the largest value that is not a linear combination of 7 and 6, where $w,x\geq 0$, is $7\cdot 6-7-6=29$, so the largest number that can't be written in the form $14w+12x$ $=2(7w+6x)$ is $2\cdot 29=58$.

19. $2^2-1=3$, which is prime. $2^3-1=7$, which is prime. $2^5-1=31$, which is prime. $2^7-1=127$, which is prime. $2^{11}-1=2047=23\cdot 89$, so 11 is the first such prime.

20. $(2n^2+5n+1)+((n+1)^2+2(n+1)+1)=4n^2+n+5 \Rightarrow 3n^2+9n+5=4n^2+n+5$ $\Rightarrow 0=n^2-8n=n(n-8)\Rightarrow n=0$ or $n=8$, but $n=8$ is the only answer that makes sense.

21. Since dividing 7's out of 2011! would not take out any of the factors of 10, we must make the number of 7's one larger than the number of 7's that appear in the factorization of 2011!, thus making x a non-integer. The number of 7's in the factorization of x is $\left\lfloor \frac{2011}{7} \right\rfloor+\left\lfloor \frac{2011}{7^2} \right\rfloor+\left\lfloor \frac{2011}{7^3} \right\rfloor+\left\lfloor \frac{2011}{7^4} \right\rfloor+...=287+41+5+0+...$ $=333$, where the ... represents infinitely many 0's being added to the sum. Therefore, the smallest n should be $333+1=334$.

22. The sum of the digits of the number is $1+2+3+A+7+8+2+B=23+A+B$, so be divisible by 3, we must have $A+B\equiv 1(\text{mod}3)$. To make the number divisible by 2, we must have B be an even digit. Therefore, B is 0, 2, 4, 6, or 8. If $B=0$, A could be 1, 4, or 7; if $B=2$, A could be 2, 5, or 8; if $B=4$, A could be 0, 3, 6, or 9; if $B=6$, A could be 1, 4, or 7; and if $B=8$, A could be 2, 5, or 8. Therefore, there are a total of 16 possible ordered pairs.

23. The number of positive integral divisors divisible by 2 is $a(b+1)(c+1)$ $=abc++ab+ac+a$. The number of positive integral divisors divisible by 3 is $(a+1)b(c+1)=abc+ab+bc+b$. The number of positive integral divisors divisible by 5 is $(a+1)(b+1)c=abc+ac+bc+c$. Therefore, $x+y+z=3abc+2ab+2ac+2bc$ $+a+b+c$.

24. $210102_3=712_9$ and $120032_4=3016_8$, so the base-10 sum of the digits of these two numbers is $7+1+2+3+0+1+6=20$.

25. We must remove any integers divisible by 2 or 3 from the list of 99 integers. The number of integers divisible by 2 is $\left\lfloor \frac{99}{2} \right\rfloor = 49$, and the number of integers divisible by 3 is $\left\lfloor \frac{99}{3} \right\rfloor = 33$, but we have double counted the integers divisible by both 2 and 3, namely those divisible by 6, of which there are $\left\lfloor \frac{99}{6} \right\rfloor = 16$. Thus, there are a total of $49 + 33 - 16 = 66$ to be removed from the list of 99, leaving us with a total of 33.

26. Using Fermat's Little Theorem, since 23 and 43 are relatively prime, $23^{42} \equiv 1 \pmod{43} \Rightarrow 23^{43} \equiv 23 \pmod{43}$, so the answer is 23.

27. Let F be the sought limit. Then $F = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{F} \Rightarrow F^2 = F + 1$ $\Rightarrow 0 = F^2 - F - 1 \Rightarrow F = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$, but the limit must be positive since the terms are positive as $n \to \infty$. Therefore, $F = \frac{1+\sqrt{5}}{2}$.

28. The acceptable representations are: for triangular numbers, $1+1+6$; for square numbers, $4+4$; and for pentagonal numbers, $5+1+1+1$. The sum of the squares of these digits is $1^2 + 1^2 + 6^2 + 4^2 + 4^2 + 5^2 + 1^2 + 1^2 + 1^2 = 1+1+36+16+16+25+1+1$ $+1 = 98$.

29. The next smaller ordered pair is $(17,12)$, and by the last sentence, the approximation for $\sqrt{2}$ would be x/y, since $x^2/y^2 = 2$. This value is $17/12 = 1.416666....$, which when rounded to five decimal places is 1.41667.

30. $14400 = 2^6 3^2 5^2$, so x^2 must be selected from this, meaning x must be selected from $2^3 3^1 5^1$, which has a total of $4 \cdot 2 \cdot 2 = 16$ positive integral divisors. Since picking each x automatically picks the y to go with it, there are 16 ordered pairs.