Answers:

Relay 1
1. 11
2. 8
3. 1080
4. 32
5. 2
6. 7

Relay 2
1. 16
2. 65
3. 1073
4. 2148
5. 24
6. 576

Relay 3
1. 35
2. 21
3. 39
4. 9
5. 257
6. 4

Relay 4
1. 169
2. 37
3. 111
4. 12
5. 21
6. 48

Relay 5
1. 306
2. 6
3. 10
4. 20
5. 99
6. 9

Solutions:

Relay 1

1. $\left\|\langle -6,-7,6\rangle\right\| = \sqrt{(-6)^2+(-7)^2+6^2} = \sqrt{36+49+36} = \sqrt{121} = 11$

2. $(11+1)^4 = 12^4 = 4^4 \cdot 3^4 = 2^8 \cdot 3^4$, so the answer is 8

3. $180^\circ(8-2) = 180^\circ(6) = 1080^\circ$

4. $1080 = 2^3 \cdot 3^3 \cdot 5$, so there are $(3+1)(3+1)(1+1) = 4\cdot 4\cdot 2 = 32$ positive integral factors

5. $N = 3+2 = 5$, so the real 5th root of 32 is 2

6. $$\begin{aligned}A(2,2) &= A(1,A(2,1)) = A(1,A(1,A(2,0))) = A(1,A(1,A(1,1)))\\
&= A(1,A(1,A(0,A(1,0)))) = A(1,A(1,A(0,A(0,1)))) = A(1,A(1,A(0,2))) = A(1,A(1,3))\\
&= A(1,A(0,A(1,2))) = A(1,A(0,A(0,A(1,1)))) = A(1,A(0,A(0,A(0,A(1,0)))))\\
&= A(1,A(0,A(0,A(0,A(0,1))))) = A(1,A(0,A(0,A(0,2)))) = A(1,A(0,A(0,3)))\\
&= A(1,A(0,4)) = A(1,5) = A(0,A(1,4)) = A(0,A(0,A(1,3))) = A(0,A(0,A(0,A(1,2))))\\
&= A(0,A(0,A(0,A(0,A(1,1))))) = A(0,A(0,A(0,A(0,A(0,A(1,0))))))\\
&= A(0,A(0,A(0,A(0,A(0,A(0,1)))))) = A(0,A(0,A(0,A(0,A(0,2)))))\\
&= A(0,A(0,A(0,A(0,3)))) = A(0,A(0,A(0,4))) = A(0,A(0,5)) = A(0,6) = 7\end{aligned}$$

Relay 2

1. $\pi/4 < \theta/2 < \pi/2$, so $\cos(\theta/2) > 0$, and $\cos(\theta/2) = \sqrt{\dfrac{1+\cos\theta}{2}} = \sqrt{\dfrac{1+\left(-127/128\right)}{2}}$
$= \sqrt{\dfrac{1/128}{2}} = \sqrt{1/256} = 1/16$, so $\sec(\theta/2) = 16$

2. $\sqrt{16^2+(-63)^2}=\sqrt{256+3969}=\sqrt{4225}=65$

3. $P(\le 2\text{ heads})=\dfrac{\binom{65}{0}+\binom{65}{1}+\binom{65}{2}}{2^{65}}=\dfrac{1+65+2080}{2^{65}}=\dfrac{2146}{2^{65}}=\dfrac{1073}{2^{64}}$, so the answer is 1073

4. $\begin{bmatrix}1&2\\0&1\end{bmatrix}^n=\begin{bmatrix}1&2n\\0&1\end{bmatrix}$, so $\begin{bmatrix}1&2\\0&1\end{bmatrix}^{1073}=\begin{bmatrix}1&2146\\0&1\end{bmatrix}$, so the sum of the entries is $1+2146+0+1=2148$

5. $T=2+1+4+8=15$. If a and b are the two digits, then $10a+b=3ab$, so $b=\dfrac{10a}{3a-1}$. The only solutions in integers to this equation are $a=1,b=5$ and $a=2,b=4$. So the answer is 24.

6. There are 24 possible values for both $f(0)$ and $f(1)$, so there are $24^2=576$ different functions.

Relay 3

1. The remainder is $(-2)^5+12(-2)^4+3(-2)^3-20(-2)^2+13(-2)+5$
$=-32+192-24-80-26+5=35$

2. $9a+3b+c=57$, $4a+2b+c=35$, and $a-b+c=29\Rightarrow 5a+b=22$ and $8a+4b=28$
$\Rightarrow 5a+b=22$ and $2a+b=7\Rightarrow 3a=15\Rightarrow a=5\Rightarrow b=-3\Rightarrow c=21$

3. $\dfrac{12x^2+27x+21}{x^3-x^2+4x-4}=\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x-1}=\dfrac{(A+C)x^2+(B-A)x+(4C-B)}{x^3-x^2+4x-4}$, so $A+C=12$, $B-A=27$, and $4C-B=21\Rightarrow B+C=39$ and $4C-B=21\Rightarrow 5C=60\Rightarrow C=12$
$\Rightarrow B=27\Rightarrow A=0$. Therefore, $A+B+C=0+27+12=39$.

4. $12+3\cdot 9=39$, so the score is 9 standard deviations away from the mean

5. The sequence is $2,3,5,9,17,33,65,129,257,\ldots$, so the 9th term is 257

6. For $\log_a 256$ to be an integer, a must be a power of 2. The only values that work are 2, 4, 16, and 256, so there are 4 such integers.

Relay 4

1. $\left|(3-2i)^4\right| = \left(\sqrt{13}\right)^4 = 169$

2. The sequence is 2011, 6, 36, 45, 41, 17, 50, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58,…, and 58 yields the same successor as 85. Therefore, this sequence has 10 initial terms, then the next 8 terms that repeat over and over. $169 = 10 + 8\cdot 19 + 7$, so the 169th term is the same as the 7th term in the repeating cycle, which is 37.

3. $\frac{1}{3}\pi(3)^2 \cdot 37 = 111\pi$, so the answer is 111

4. $111^2 = 12321$, so $P(12321) = 1\cdot 2\cdot 3\cdot 2\cdot 1 = 12$

5. $\frac{1}{A} + \frac{1}{B} = \frac{1}{12} \Rightarrow \frac{A+B}{AB} = \frac{1}{12}$. Setting $A+B=k$ and $AB=12k$, A and B are the solutions to the equation $x^2 - kx + 12k = 0$. Those solutions are $x = \frac{k \pm \sqrt{k^2 - 4(1)(12k)}}{2(1)} = \frac{k \pm \sqrt{k^2 - 48k}}{2}$, and k is as small as possible. To make these solutions integers, $k^2 - 48k \geq 0$ must be a perfect square. The smallest such value of k is 48, but that makes $A = B = 24$, contradicting $A < B$. The next larger value of k is 49, making $A = 21$ and $B = 28$. So the answer is 21.

6. $3(69) - 159 = 207 - 159 = 48$

Relay 5

1. The only two right triangles with hypotenuse 25 are 7-24-25 and 15-20-25. If 25 is a leg length, $c^2 - b^2 = 25^2 \Rightarrow (c-b)(c+b) = 25^2$. The only possibilities are $c + b = 625$ and $c - b = 1 \Rightarrow c = 313$ and $b = 312$ or $c + b = 125$ and $c - b = 5$ $\Rightarrow c = 65$ and $b = 60$. Of the four triples, the smallest side is 7 and the largest side is 313. So $313 - 7 = 306$.

2. $x^2 - 57x + 306 = (x-6)(x-51)$, so the smaller root is 6

3. The major axis has length 22 and the minor axis has length 8. Therefore, the foci are of length $\sqrt{11^2 - 4^2} = \sqrt{121-16} = \sqrt{105}$, and $\left\lfloor \sqrt{105} \right\rfloor = 10$.

4. $y=\sin(10x)$ crosses the x-axis twice every $\pi/5$, without counting the right endpoint of each of those subintervals. Therefore, the graph intersects 20 times.

5. $\left\lfloor 400/5 \right\rfloor + \left\lfloor 400/25 \right\rfloor + \left\lfloor 400/125 \right\rfloor = 80+16+3=99$

6. $\binom{99}{2}=4851$, so $P(4851)=R(4+8+5+1)=R(18)=1+8=9$