The following were changed at the resolution center at the convention: 10 B or C, 28 E

1. There are two ways to solve this. One *could* square each number by hand and check. However, the shorter method would be to work in modulo 10. For D, this would imply that $6^2 + 8^2 \equiv 2^2$ which is false. The other choices pass this first test, hence, the answer is **D**

2. 24! has factors of 5, 10, 15, 20 which all have a factor of 5; hence 24! is divisible by 10^4 but not 10^5. 25! has factors 25, 20, 15, 10, 5. Since $25 = 5^2$, that means 25! has 6 factors of 5. Hence there is no number whose factorial meets the given conditions and the correct answer is **E**

3. $224 = 32*7 = 2^5 * 7$ so any number of the form $2^x * 7^y$ is a factor of 224, where $0 \le x \le 5$ and $0 \le y \le 1$. So there are two choices for y, 6 choices for x. 12 factors total, but 224 is not a *proper* divisor of itself, so 11 proper divisors. **C**

4. Any perfect square must be ≡ 0 or 1 (mod 4), so B is true. Since $k = 4n + 3$, k is odd, so A is true. C is not true; the correct combination would be $k \equiv 23 \pmod{28}$ so the correct answer is **D**

5. Clearly gcd(11,37) = 1 since both are prime. Using the Euclidean algorithm, we get that for $37a + 11b = \gcd(11,37) = 1$, $a = 3$ and $b = -10$. We could multiply this by 3 and use the pair (9, -10) but 9 is not a given answer. Instead, note that an intermediate step of the Euclidean algorithm yields the pair (-2, 7). **B**

6. $\varphi(36) = \varphi(2^2 * 3^2) = \varphi(2^2)*\varphi(3^2) = (2^1)(2-1)(3^1)(3-1) = 2*1*3*2 = 12$ **A**

7. $\varphi(7) = 6$, $\varphi(13) = 12$, $\varphi(19) = 18$, $\varphi(35) = 24$. Other solutions exist; these are the smallest x. 14 and 26 are nontotient. **B**

8. A is infinite (known since Euclid, at least). B is infinite; since it is not specified that they be primitive, we could take for example all integral multiples of (3,4,5). D is infinite, if we consider only polygons with 2^n sides by constructing a square, and bisecting central angles. C may be infinite, but this is unknown (Twin Prime Conjecture) so the answer is **C**

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9. The sum of the digits 1+5+1+4+7 = 18, so 15147 is divisible by 9; 15147 = 9*1683. Sum of digits of 1683 is 18, so divide again by 9, $=3^4 * 187$. 187 = 11*17 (if this isn't immediately clear, try the mod 11 test of 1+8-7 = 0) so $15147 = 3^4 * 11 * 17$. 3+11+17 = 31 **D**

10. If k is prime, then k^2 has two proper divisors, 1 and k. If k is a perfect square, then k^2 has an odd number of proper divisors. If $k = p^3$, then $k^2 = p^6$ has proper divisors 1, p, … through p^5, so there are 6 proper divisors. If k has 4 proper divisors, then k^2 has at least 8 proper divisors, just counting the divisors of k and their squares. The answer is **C**

11. 8 cannot be expressed as the sum of two cubes (as defined in the start of the test). $72 = 4^3 + 2^3$ which is only one sum. $1216 = 10^3 + 6^3$, similarly only one way to do so. $1729 = 1^3 + 12^3 = 10^3 + 9^3$ so the correct answer is **B**

12. By Fermat's Little Theorem, 23 is prime so 11^{22} is congruent to 1. **A**

13. 191 is prime, so the N = 1 and M = 191*214 = 40874. M – N = 40873 **D**

14. There are 11 partitions. In a slightly compressed list form, they are: 6, 15, 114, 24, 33, 1113, 123, 222, 1122, 11112, 111111 **E**

15. I may be true in certain cases (for example, 15 divides 14! since 5 and 3 and both divisors of 14!) but is not true in general (for example, whenever n+1 is prime). II is true because n+1 includes (but is not limited to) any partition of the form {1, X} where X is any partition of n. III is the Bertrand-Chebyshev Theorem, which is true only for $n \geq 2$, a condition we do have here. II and III are correct, so the answer is **E**

16. There are x = 7 partitions of 5 and y = 5 partitions of 4. 7 (mod 5) = 2 **B**

17. Solving the second and third congruences gives $x \equiv 56 \pmod{143}$; adding in the first congruence we get $x \equiv 485 \pmod{1001}$. 1001 / 485 = 2.060, so the closest integer is 2 **D**

18. By Fermat's Little Theorem, $13^{30} \equiv 1 \pmod{31}$, so $13^{31} \equiv 13 \pmod{31}$ **C**

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19. A = gcd(7*12, 3*12) = 12; B = lcm(14, 14*7) = 98; C = 120-5 = 115; D ≡ 4 (mod 7), but D is not used in the expression. The expression is thus (12+98)(115-12) = (110)(103) = 11000 + 330 = 11330 **A**

20. The golden ratio $\varphi = (1+\sqrt{5}) / 2$, the solution to the equation $x^2 - x - 1 = 0$ **C**

21. Algebraic numbers are closed under addition and multiplication, so I and III are algebraic. Logarithms are not necessarily algebraic, and neither are exponents. **C**

22. If we have a polynomial in the rationals, we can turn it into a polynomial with integer coefficients by multiplying by the LCM of all denominators of fractional coefficients. Hence, α is certainly algebraic since it will also be a solution to this integer polynomial. **A**

23. Long division could be used here, but the shortcut is to sum up the digits; if the sum of the digits of a number is divisible by 9 then the number itself is. Number A sums to 40. Number B sums to 61. Number C sums to 39. Number D sums to 37. None of them are divisible by 9, so the correct answer is **E**

24. 8741 = 2(4096) + 1(512) + 0(64) + 4(8) + 5(1) so 21045 **B**

25. $135 = (3^3)(5)$. Since both prime factors are odd, any factor of 135 is odd, so probability of choosing an odd factor is = 1 **E**

26. There are 25 prime numbers less than 100; however Sven will not step on the number 3, since 2 is prime he skips over 3. Thus he skips 24 steps, and stands on the other 76. **D**

27. Sven's path downstairs is the following (primes in bold): 100, 99, 98, **97**, 87, 86, 85, 84, **83**, **73**, 64, 63, 62, **61**, **53**, 45, 44, **43**, 36, 35, 34, 33, 32, **31**, 25, 24, **23**, 18, **17**, 12, **11**, **7**, 4, **3**, 1. He touches 12 prime-numbered steps. **B**

28. Reals and positive integers are closed under exponentiation; this should be obvious. The set in IV is also obviously closed. Rationals are not, because for example $2^{1/2} = \sqrt{2}$ is not rational **D**

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29. The Fibonacci sequence is, in part: 0, 1, 1, 2, 3, 5, 8, … The sum of the first 6 is 0+1+1+2+3+5 = 12 (Or a shortcut, the sum is one less than the next Fibonacci number, in this case 5+8=13) **D**

30. Gaussian integers (and all complex numbers in general) retain commutativity and associativity of multiplication. Unique factorization holds for Gaussian integers; non-unit elements factor uniquely into products of irreducible elements. Not that a counterexample such as 10 = (2)(5) = (3 + *i*)(3 – *I*) is not valid, since 2 = (1+*i*)(1-*i*) so 2 is not irreducible in Z[*i*]. *Z* is totally ordered, whereas any ordering of Z[*i*] does not preserve the relevant arithmetic. **D**