Logs, Exponents, Radicals - Alpha Solutions MAΘ National Convention 2010

The following were changed at the resolution center at the convention: 16 E

1. D. $\left(\sqrt[3]{256}-\sqrt[3]{108}\right)^{-3/2}=\left(4\sqrt[3]{4}-3\sqrt[3]{4}\right)^{-3/2}=\left(2^{2/3}\right)^{-3/2}=\frac{1}{2}$

2. A $\log(ab)+\log\left(\frac{1}{ab}\right)=\log 1=0=\log_2 r \text{ iff } r=1$

3. E $\begin{cases}\log_8 1+\log_3(x+2)=\log_3(3-2y)\\ 2^{x+y}-8^{3-y}=0\end{cases}\Rightarrow \begin{matrix}x+2=3-2y\\ x+y=9-3y\end{matrix}\Rightarrow$ $L_1-L_2,\ 2-y=-6+y$, $8=2y$, $y=4,\ x=-7$, not soln for log.

4. C $\frac{1}{\log_4 18}+\frac{1}{2\log_6 3+\log_6 2}+\frac{5}{\log_3 18}=\frac{\log 4}{\log 18}+\frac{\log 6}{\log 18}+\frac{\log 3^5}{\log 18}=\frac{\log(4\cdot 6\cdot 243)}{\log 18}=\log_{18}18^3=3$

5. C $\log_2\left[\left(\sqrt{7}-\sqrt{5}\right)\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}\right)\right]=\log_2\frac{2}{\sqrt{7}+\sqrt{5}}=\log_2 2-\log_2\sqrt{7}+\sqrt{5}=1-S$

6. C $x^2-2xy=0$ and $x^2=y+3$

$\Rightarrow x(x-2y)=0$

$\Rightarrow x\neq 0,\ x=2y\Rightarrow(2y)^2=y+3\Rightarrow 4y^2-y-3=0$

$\Rightarrow(4y+3)(y-1)=0\Rightarrow y=1, x=2\ (x>0\Rightarrow y>0)$

7. D $\log_b 125=3\log_b 5=c;\ \frac{2\log_b 5}{3\log_b 5}=\frac{2}{3}c\Rightarrow 66\,{}^2\!/_3\%$ of c.

8. B $\log_x x^{x^2}+\log_x x^{-5x}=\log_x\left(\frac{1}{x^6}\right)\Rightarrow\left(x^{x^2}\right)\left(x^{-5x}\right)=x^{-6}\Rightarrow$

$x^{x^2-5x}=x^{-6}\Rightarrow x^2-5x+6=0\Rightarrow(x-2)(x-3)=0;\ x=2, 3$

9. D $\ln(\sin x)-\ln(\cos x)=1\Rightarrow \tan x=e\Rightarrow x=\arctan e$

10. D $\ln(y-2)=\ln(\sin x)-3x\Rightarrow 3x=\ln(\sin x)-\ln(y-2)$

$\Rightarrow e^{3x}=\frac{\sin x}{y-2}\Rightarrow\frac{\sin x}{e^{3x}}=y-2\Rightarrow y=e^{-3x}\sin x+2$

11. D $\log_3 x=2\cos x\to\log_3 9=2$, the max val of $2\cos x$. For $x>9$, $\log_3 x>2$ hence 3 points

12. B. $3^{\tan x}=27^{\sin x}\Rightarrow\tan x=3\sin x\Rightarrow\sin x=3\sin x\cos x\Rightarrow\sin x(1-3\cos x)=0.\ \cos x=\frac{1}{3}$ or 1 in the given domain.

13. C $4^x-3^2\cdot 2^x+2^3=0\Rightarrow 2^{2x}-9\cdot 2^x+8=0\Rightarrow(2^x-8)(2^x-1)=0\Rightarrow x=3, 0\Rightarrow \text{sum}=3$

14. B $f(x)=\log(x+2)^3\to x=3\log(f^{-1}+2)\Rightarrow 10^{x/3}=f^{-1}+2\Rightarrow f^{-1}=\sqrt[3]{10^x}-2$

15. A $\left(\frac{54}{111}\right)_x = \left(\frac{16}{25}\right)_x \Rightarrow \frac{5x+4}{x^2+x+1} = \frac{x+6}{2x+5} \Rightarrow 10x^2+33x+20 = x^3+7x^2+7x+6$

$\Rightarrow x^3-3x^2-26x-14=0$

$\Rightarrow (x-7)(x^2+4x+2)=0;\quad x=7 \text{ only integer soln.}$

16. C $\frac{\log_a(x-3)\cdot\log_b(x+10)}{\log_b(x-3)} = 2 \Rightarrow \frac{\log(x-3)\cdot\log(x+10)\cdot\log(x-4)}{\log(x-2)\cdot\log(x-4)\cdot\log(x-3)} = \log_{(x-2)}(x+10) = 2$

$\Rightarrow x^2-4x+4 = x+10 \Rightarrow x^2-5x-6=0$

$\Rightarrow (x-6)(x+1)=0;\ x\neq -1,\ \therefore x=6$

17. A $\sqrt{2x} = \sqrt{x+7} - 1 \Rightarrow 2x = x+7-2\sqrt{x+7}+1 \Rightarrow x-8 = -2\sqrt{x+7} \Rightarrow x^2-16x+64 = 4x+28$

$\Rightarrow x^2-20x+36=0 \Rightarrow (x-18)(x-2)=0;\ x=2,\ 18. \text{ Only } x=2 \text{ cks.}$

18. C $\frac{1}{\log_7 2} + \frac{1}{\log_9 4} = x \Rightarrow \frac{\log 7}{\log 2} + \frac{\log 9}{\log 4} \Rightarrow \frac{2\log 7}{2\log 2} + \frac{\log 9}{\log 4} \Rightarrow \frac{\log(49\cdot 9)}{\log 4} = \log_4(49\cdot 9) = x$

$\Rightarrow 4^{\log_4(49\cdot 9)} = 49\cdot 9 = 441$

19. E $\left(d^4 - \frac{2}{d^3}\right)^{15} \Rightarrow \binom{15}{5}(d^4)^5(-2d^{-3})^{10} = (3003)(1024)d^{-10}$

20. D $\begin{cases} \log_x 9 + \log_8 y = \frac{7}{3} \\ \log_9 x + \log_y 8 = \frac{7}{2} \end{cases}$; Let $\begin{cases} a = \log_9 x \\ b = \log_8 y \end{cases} \Rightarrow \begin{cases} \frac{1}{a} + b = \frac{7}{3} \\ a + \frac{1}{b} = \frac{7}{2} \end{cases} \Rightarrow \begin{cases} \\ a = \frac{7}{2} - \frac{1}{b} = \frac{7b-2}{2b} \end{cases}$

$\Rightarrow L_1 \rightarrow \frac{3(2b)}{7b-2} + 3b = 7 \Rightarrow 6b = (7-3b)(7b-2)$

$\Rightarrow 21b^2-49b+14=0;\ b=\frac{1}{3},\ 2 \text{ hence } a=\frac{1}{2},\ 3$

hence $(x,\ y) = (3,\ 2)$ or $(729,\ 64)$

21. E $f(x) = \sqrt{\frac{4}{x-2}}$ real only $\frac{4}{x-2} \geq 0,\ x-2 \neq 0 \Rightarrow x > 2$

22. B $\ln(\log_7 35 - \log_7 5) = \ln\left(\log_7 \frac{35}{5}\right) = \ln 1 = 0$

23. C $\left(2^{1/3}\right)\left(4^{1/9}\right)\left(8^{1/27}\right)\left(16^{1/81}\right)\cdots = \left(2^{1/3}\right)\left(2^{2/9}\right)\left(2^{3/27}\right)\left(2^{4/81}\right)\cdots = 2^{1/3+1/9+1/9+1/27+1/27+1/27+1/81+1/81+1/81+1/81+\ldots}$

Using partial sums of inf. geom. series in the exponent of 2:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots=\frac{1/3}{1-1/3}=\frac{1}{2}$$

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots=\frac{1/9}{1-1/3}=\frac{1}{6}$$

$$\frac{1}{27}+\frac{1}{81}+\ldots=\frac{1/27}{1-1/3}=\frac{1}{18} \quad\Rightarrow\quad \frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\ldots=\frac{1/2}{1-1/3}=\frac{3}{4}\Rightarrow \text{Product}=2^{3/4}$$

24. A $\begin{cases} 2^{x-y}-8^{3-y}=0 \\ \log_3 1+\log_5(x+2)=\log_5(2y-3)\end{cases} \Rightarrow \begin{cases} x-y=9-3y \\ x+2=2y-3\end{cases} \Rightarrow L_2-L_1\ \ 2+y=5y-12 \Rightarrow 14=4y,\ \begin{matrix} y=3.5 \\ x=2\end{matrix}$

25. B $\left((2a-3b)+c\right)^6 \Rightarrow \binom{6}{3}(2a-3b)^3c^3=20c^3(2a-3b)^3$; Sum of terms $20(-1)=-20$

26. C $\log_4(3x+7)-\log_4(x-5)=2\Rightarrow \frac{3x+7}{x-5}=4^2=16\Rightarrow 3x+7=16x-80\Rightarrow x=\frac{87}{13}>6$

27. D Inf. geom series, $\sum_{k=1}^{\infty}\frac{3\left(4^{k-1}\right)}{5^k}=\frac{3/5}{1-4/5}=3$

28. B $a^{2b}=5,\ \ 2a^{6b}-4=2(5)^3-4=246$

29. A $\frac{e^x+e^{-x}}{e^x-e^{-x}}=2\Rightarrow e^x+e^{-x}=2e^x-2e^{-x}\Rightarrow 3e^{-x}=e^x\Rightarrow 3=e^{2x}\Rightarrow 2x=\ln 3\Rightarrow x=\frac{1}{2}\ln 3$

30. B $\frac{2^{-1}\left(16x^5\right)^{1/4}}{2x^{1/4}}=\frac{2x^{5/4}}{4x^{1/4}}=\frac{x}{2}$