On this test, we will let $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{Z}$ = the set of integers, $\mathbb{Q}$ = the set of rational numbers, $\mathbb{R}$ = the set of real numbers, and $\mathbb{C}$ = the set of complex numbers. We define the imaginary unit as $i = \sqrt{-1}$, and if $z = a + bi$ then $\Re(z) = a$ (the real part) and $\Im(z) = b$ (the imaginary part). You may assume that $a, b \in \mathbb{R}$ unless otherwise specified. Finally, $\overline{z}$ represents the complex conjugate of z. Take a deep breath, enjoy, and good luck!

1. Simplify $2i(3 - 4i) - (1 - i)(2 + 5i)$

   A. $1 + 9i$ B. $-5 + 3i$ C. $-5 + 9i$ D. $1 + 3i$ E. NOTA

2. If $w = a + bi$ and $z = c + di$, find $\overline{wz}$.

   A. $(ac + bd) + (ad - bc)i$ B. $(ac - bd) + (ad + bc)i$ C. $(ac + bd) - (ad - bc)i$

   D. $(ac - bd) - (ad + bc)i$ E. NOTA

3. Evaluate $(1 - 2i)^4$.

   A. $-7 + 24i$ B. 15 C. $25 + 24i$ D. $41 - 40i$ E. NOTA

4. Find the multiplicative inverse of $3e^{i\theta}$ in vector form.

   A. $\langle 3\cos(\theta), 3\sin(\theta)\rangle$ B. $\langle 3\cos(\theta), -3\sin(\theta)\rangle$ C. $\left\langle \frac{\cos(\theta)}{3}, -\frac{\sin(\theta)}{3} \right\rangle$

   D. $\left\langle -\frac{\cos(\theta)}{3}, \frac{\sin(\theta)}{3} \right\rangle$ E. NOTA

5. Find and simplify the complex conjugate of $\frac{3}{1 - 4i}$.

   A. $\frac{3}{17} + \frac{12}{17}i$ B. $\frac{3}{17} - \frac{12}{17}i$ C. $\frac{12}{17} - \frac{3}{17}i$ D. $\frac{12}{17} + \frac{3}{17}i$ E. NOTA

6. Find the number of real zeros of the function $f(z) = z^5 + (5 - 2i)z^4 - (3 + 10i)z^3 + (6i - 15)z^2 + 30iz$.

   A. 1 B. 2 C. 3 D. 5 E. NOTA

7. Let $z = a + bi$. Find $a + b$ when $|\overline{z}| = 2\sqrt{5}$ and $\frac{25}{z} - \frac{15}{\overline{z}} = 1 - 8i$.

   A. $-\frac{7}{2}$ B. -2 C. $\frac{9}{2}$ D. 6 E. NOTA

8. Find $\sqrt[\pi]{(-1)^{-i}}$

   A. $e^{-3/2}$ B. e C. e^{-1} D. 1 E. NOTA

9. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_5 \neq 0$ and $a_0, ..., a_5 \in \mathbb{C}$. Now suppose that $x - i$ is a factor of $f(x)$. What is the minimum number of real zeros this function could have?

A. 0 B. 1 C. 3 D. 4 E. NOTA

10. Let $f(z) = \dfrac{1}{1-|z|^2}$, with $z \in \mathbb{C}$. Find all points not in the domain of f.

A. $\varnothing$ B. $\{-1, 1, -i, i\}$ C. $\{-1, 1\}$ D. $\{e^{i\theta} \in \mathbb{C} | \theta \in [0, 2\pi)\}$
E. NOTA

11. Find i^n when $n = 7^{127}$.

A. 1 B. -1 C. i D. $-i$ E. NOTA

12. Which of the following are true for $z \in \mathbb{C}$?

I. $|\cos(z)|$ is bounded; that is, there is a real number M such that for every $z \in \mathbb{C}$, $|\cos(z)| \leq M$.
II. Geometrically, multiplying two complex numbers multiplies lengths and adds angles.
III. The natural logarithm cannot be defined for complex z.

(A) II only (B) III only (C) I, II (D) I, II, III (E) NOTA

13. Evaluate: $\arcsin(\cosh(i\frac{4\pi}{3}))$.

(A) $-\dfrac{5\pi}{6}$ (B) $-\dfrac{\pi}{6}$ (C) $\dfrac{5\pi}{6}$ (D) $\dfrac{7\pi}{6}$ (E) NOTA

14. Which of these is a possible value of $\ln(-ei)$?

(A) $1 + \dfrac{\pi}{2}i$ (B) $-1 + \dfrac{3\pi}{2}i$ (C) $1 - \dfrac{\pi}{2}i$ (D) $(\pi - 1)i$ (E) NOTA

15. Which of the following theorems is not a theorem of complex variables?

(A) The pigeon-hole principal (B) Picard's Big Theorem (C) Liouville's Theorem
(D) Fundamental theorem of Algebra (E) NOTA

16. Evaluate: $\cos(3i\ln(i))$.

(A) DNE (B) 1 (C) 0 (D) -1 (E) NOTA

17. For how many values of z does z equal its multiplicative inverse?

(A) 0 (B) 1 (C) 2 (D) 4 (E) NOTA

18. For how many values of z does z equal the reciprocal of its conjugate?

(A) 0 (B) 1 (C) 2 (D) 4 (E) NOTA

19. Let the following points in the complex plane lie on a circle, C: $-2+i, 1+4i, 4+i$. The modulus of the center falls in which of the following intervals?

(A) $[0,1)$ (B) $[1,2)$ (C) $[2,3)$ (D) $[3,4)$ (E) NOTA

One can define the complex numbers as follows: $\mathbb{C} = \{z = a+bi \mid a,b \in \mathbb{R}, i^2 = -1\}$**. The *hypercomplex numbers* or Quaternions are defined as follows:**

$$\mathbb{H} = \{w = a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

Also, $ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$**, and the left- and right-distributive properties hold.**

20. Which of the following are true about the Quaternions?

I. $\mathbb{H}$ is commutative under addition
II. $\mathbb{H}$ is commutative under multiplication
III. $\mathbb{H}$ is associative under addition

(A) I only (B) I,II (C) I, III (D) I, II, III (E) NOTA

21. If $w = 3i - k$, and $z = 5+3j+2k$, then $z \cdot w$ will have the form $a+bi+cj+dk$. Using the rules given above, find $a+b+c+d$.

(A) 2 (B) 6 (C) 26 (D) 30 (E) NOTA

22. The Quaternions were discovered by a mathematician whose last name begins with H, which is why we denote the set of Quaternions by $\mathbb{H}$. Which of the following distinguished mathematicians discovered the Quaternions while on an afternoon stroll with his lady?

(A) David Hilbert (B) Felix Hausdorff (C) Paul Halmos (D) William Hamilton

(E) NOTA

23. If $\Im(z) = 6$ and $|\overline{z}| = 8$, which of the following is a possible value of $\Re(z)$?

(A) -2 (B) $\sqrt{58}$ (C) 2 (D) $-4\sqrt{7}$ (E) NOTA

24. What is the shape of the locus of points represented by $z + \overline{z} = 2$ in the complex plane?

(A) horizontal line (B) vertical line (C) circle (D) hyperbola (E) NOTA

25. Find $\left|\dfrac{3+2i}{1-i}\right|$. The solution can be written in lowest terms in the form $\dfrac{p}{q}$ where $p \in \mathbb{R}$ and and $q \in \mathbb{N}$. Find p.

(A) $\sqrt{13}$ (B) $\sqrt{2}$ (C) $\sqrt{26}$ (D) 2 (E) NOTA

26. How many complex numbers can be written in the form $z = p + iq$ where $p, q \in \mathbb{Z}$ and $|\overline{z}| < \sqrt{6}$?

(A) 8 (B) 26 (C) 16 (D) 21 (E) NOTA

27. Let a_n be the n^{th} term of a complex-valued geometric sequence such that $a_2 = -3i$ and $a_5 = 24$. Which of the following could be a_7?

(A) $-48 - i48\sqrt{3}$ (B) 96 (C) $48 - i48\sqrt{3}$ (D) $-48i$ (E) NOTA

28. Evaluate $|5 - 3i|$.

(A) $\sqrt{34}$ (B) 4 (C) 2 (D) $\sqrt{10}$ (E) NOTA

29. In which quadrant does $(2 - i)(2 + i)$ fall?

(A) I (B) II (C) III (D) IV (E) NOTA

30. Find $\Im(w)$ when $w = \left(\dfrac{1}{2} + i\dfrac{1}{\sqrt{2}}\right)^5$.

(A) 1 (B) $\dfrac{\sqrt{2}}{4}$ (C) $-\dfrac{1}{4}$ (D) $-\dfrac{1}{2\sqrt{2}}$ (E) NOTA

TB1 Indicate, in set notation, all values of $z \in \mathbb{C}$ for which z equals the negative reciprocal of its conjugate.

TB2 Evaluate $i^{4\pi/i}$.

TB3 Let $f(x)$ be a polynomial of degree n with real coefficients. Suppose now that I know of k complex zeros of f, *none of which are conjugates of one another*. Write an expression that indicates the maximum number of real zeros that this polynomials can have, in terms of n and k.