2009 Number Theory (Open)

Solutions

1. $5^{12} - 4^{12} \to (5^6 + 4^6)(5^6 - 4^6) \to (5^2 + 4^2)(5^4 - 5^2 4^2 + 4^4)(5^3 - 4^3)(5^3 + 4^3) \to$
 $41(625 + 256 - 400)(5 - 4)(5^2 + 20 + 4^2)(5 + 4)(5^2 - 20 + 4^2) \to 41(481)(1)(61)(9)(21) \to$
 $3^3 \cdot 7 \cdot 13 \cdot 37 \cdot 41 \cdot 61 \to 41 + 61 + 37 = 139$ **A**

2. $2^4 \cdot 3 \to 48$ since $(4 + 1)(1 + 1) = 10$ **D**

3. *aaabbb* $\to a(100{,}000) + a(10{,}000) + a(1{,}000) + b(100) + b(10) + b \to a(111{,}000) +$
 $b(111) \to 111(1000a + b) \to 3 \cdot 37 \to 37$ **A**

4. $x^3 + ax^2 + 9x + 6/\ x^3 + bx^2 + 6x + 3 = (a - b)(x^2 + 3x + 3)$ and $x^3 + ax^2 + 9x + 6/(a - b)x^2 +$
 $3x + 3$ implies that $a - b = 1$ and $-a + 2b = 3 \to$ solving gives (5, 4) **B**

5. $9^{83} + 5^{32} \to 3 + 1 = 4$ **D**

9	81	729		5	25	125	625			
3	3	3		5	1	5	1			

6. $124_b + 345_b = 268_{10} \to 4b^2 + 6b + 9 - 268 = 0 \to 4b^2 + 6b - 259 = 0 \to (B - 7)(4B + 37)$
 $B = 7$ **B**

7. $(30)^4 \to 2^4 3^4 5^4 \to 5^3 - 2 = 123$ **C**

8. $\frac{72_{8!}}{18_{2!}} = \frac{72(64)(56)\ldots(24)(16)(8)}{18(16)(14)\ldots(6)(4)(2)} = 4^7$ ***E***

9. $32639 \to \frac{32639}{2} \approx 16320 \to \sqrt{16320} \approx 128$ and nearest prime is 127. by division the other
 prime is $257 \to 127 + 257 = 384$ ***A***

10. $799/5 = 155$; $799/7 = 113$; $799/35 = 22 \to 268 - 22 = 246$; $799 - 246 = 553$ ***C***

11. 2, 10, 30, 68, … $\to 1^3 + 1, 2^3 + 2, 3^3 + 3, 4^3 + 4, \ldots \to n^3 + n \to 20^3 + 20 = 8020$ ***A***

12. $2007^1 \to$ units digit is 7, $2007^2 \to$ units digit is 9, $2007^3 \to$ units digit is 3, $2007^4 \to$ units digit is 1,
 $2007^5 \to$ units digit is 7. $2009/4 \to$ remainder = 1 so 2007^{2009} the units digit is 7. ***D***

13. $\frac{7^{2n+1}+1}{k} \to n = 0, 8$: $n = 1, 344$, $n = 2, 16808$, $(8, 8 \cdot 43, 8 \cdot 2101\ldots) \to 8$ ***C***

14. $2a3 + 326 = 5b9 \to a + 2 = b$ and $5 + b + 9 = 18 \to b = 4$ and $a = 2 \to a + b = 6$ ***B***

15. $BA \cdot MA = QQQ \to BA \cdot MA = 111Q \to \frac{BA \cdot MA}{3 \cdot 37} = Q \to BA$ or MA are 37 or 74
 $74 \cdot 14 = 1036$ impossible so $37 \cdot 27 = 37 \cdot 3 \cdot 9$ and $T = 9 \to 3 + 7 + 2 + 9 = 21$ ***E***

16. $(2^{48} - 1) = (2^{24} + 1)(2^{24} - 1) \rightarrow (2^{12} + 1)(2^{12} - 1) \rightarrow (2^{6} + 1)(2^{6} - 1) \rightarrow 65 \cdot 63 \rightarrow 128$ **D**

17. $1441_q \rightarrow q^3 + 4q^2 + 4q + 1 \rightarrow$ implies $\frac{q^3+4q^2+4q}{11} = \frac{q(q+2)^2}{11} = \frac{11(9+2)^2}{11}$ and $q = 9$ **D**

18. $4AB = CA$ and A = 2, 4, 6, 8 but $4 \cdot 4 = 16$, $4 \cdot 6 = 24$, $4 \cdot 8 = 32$ so A must be 2 and B = 3 or 8 but $4 \cdot 28 > 99$ so $B = 3 \rightarrow 4(23) = 92$ and $C = 9$. **D**

19. ***B***

20. $2^n \cdot 3^{2n} - 1 \rightarrow 2^n \cdot 9^n - 1 \rightarrow 18^n - 1$. when $n = 2$, $18^2 - 1 = 255$, n = 3, $18^3 - 1 = 5831$ both divisible by 17. Also since $x^n - 1$ is divisible by $x - 1$ it follows that $18 - 1 = 17$ always divides $18^n - 1$. **C**

21. 309!/5 = 61, 309!/25 = 12, 309!/125 = 2. $n = 61 + 12 + 2 = 75$ **A**

22. $nA + mA + A = 30 \rightarrow A(n + m + 1) = 30$. Possible factors of 30 = 1,30..2,15..3,10..,5,6 ruling out 1,30 → $A = 2$ and $n + m = 14$; $A = 3$ and $n + m = 9$; $A = 5$, $n + m = 5$; $A = 6$, $n + m = 4$. If $A = 2$ and $m = 8$ and $n = 6$, then $2 \cdot 8 + 2 \cdot 6 + 2 = 30$.

A										
2	n,m	2,12	3,11	4,10	5,9	6,8	2+4+24	2+6+22	2+8+20	2+12+16
3		2,7	3,6	4,5			3+6+21	3+9+18	3+12+15	
5		2,3					5+10+15			
6		2,2								

(2,4,24)(2,6,22),(2,8,20)(2,10,18)(2,12,16)(3,6,21)(3,9,18)(3,12,15)(5,10,15) = 9 triples **C**

23. $14414 \cdot 14416 \cdot 14418 \rightarrow [2(7207) \cdot (7208) \cdot (7209)]/14 \rightarrow (7207) \cdot (7208) \cdot (7209)/7$
7207/7 → R4, 7208/7 → R5, 7208/7 → R6. (4)(5)(6)/14 = R8 **C**

24. $81M + 9N + P = 49P + 7N + M$; $80M + 2N = 48P$; $P = \frac{40M+N}{24} \rightarrow M = 1, N = 8, P = 2$
M = 2, N = 12, P = 4; $M = 3$, $N = 0$, $P = 5$, $N \neq 8$ (in base 7), $N \neq 12$ (In base 10). So $305_9 = 503_7$. $3 + 0 + 5 = 8$ **A**

25. Since in 2009! the number of zeros depends on the number of 5's, 2009/5 = 401, 2009/25 = 40, 2009/125 = 16, 2009/625 = 3. 401 + 40 + 16 + 3 = 460 **E**

26. ***E***

27. $7x + 11y = 100 \rightarrow x = \frac{100-11y}{7} \rightarrow (8, 4)$. 2 parts are 56 and 44 so the product = 2464 **B**

28. $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$ … cycle of 4 → 2009/4 = 502 ***E***

29. 30*AB*5 = 225*n*. 3 + 0 + *A* + *B* + 5 must be a multiple of 9 → *A* + *B* = 10. For *B* + 5 to be divisible by 25, *B* must be 2 or 7. This 30825/225 = 137 and 30375/225 = 135 **C**

30. $x + x + 1 + x + 2 + \ldots + x + 11 = 12x + 66$; $\frac{12x+66}{4} = 3x + \frac{66}{4} = 3x + 16\ R\ 2$ ***B***

Tie-Breakers:

1. 6(93) = 558 the total number of points possible. If 5(100) is highest possible for 5 tests then the 6th test must be 558 – 500 = 58.

2. $y = \frac{1-11x}{15} \rightarrow \emptyset$

3. 2009! → 460 5's so 10^{460}; $n = 460$