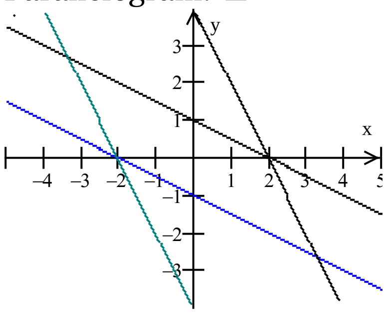
2009 Open Analytical Geometry

Solutions

1. distance = $2\sqrt{36+16} = 4\sqrt{13}$. ***B***
2. Midpoint of $EG = \left(\frac{5}{2}, \frac{3}{2}\right)$. ***D***
3. $m = \frac{3-0}{5-0} = \frac{3}{5} \rightarrow \perp -\frac{5}{3}$; $y - \frac{3}{2} = -\frac{5}{3}\left(x - \frac{5}{2}\right) \rightarrow 6y - 9 = -10x + 25$; $10x + 6y = 34$; $5x + 3y = 17$. ***C***
4. $5a + 3(a+3) = 17 \rightarrow a = 1$; $m = \frac{\frac{3}{2}-b}{\frac{5}{2}-7} = -\frac{5}{3} \rightarrow b = -6$. $a + b = -5$. ***B***
5. $A = \frac{1}{2}\begin{vmatrix} 0 & 0 \\ 1 & 4 \\ 5 & 3 \\ 7 & -6 \\ 0 & 0 \end{vmatrix} = ½|(3 - 30 - 21 - 20)| = 34$. ***C***

6. Parallelogram. ***E***



7. $r^2 = \dfrac{36}{1+8\cos^2\theta} \rightarrow x^2 + y^2 = \frac{36(x^2+y^2)}{9x^2+y^2} \rightarrow 9x^2 + y^2 = 36$. $a = 2, b = 6$. $A = 12\pi$. ***B***

8. Make vectors. Case I: initial point:(3, 7) → <3, −5> and <−1, k − 7>. Set the dot product equal to 0 to find there's only 1 solution for k.
   Case II: initial point:(6, 2). Again, only one solution for k.
   Case III: initial point: (2, k) → <1, 7 − k> and <4, 2 − k>. Letting the dot product be 0 yields two solutions for k, for a total of 4 solutions. ***D***

9. $y = r\sin\theta$, $x^2 = r^2\cos^2\theta$; $r\sin\theta = r^2\cos^2\theta \rightarrow r = \tan\theta\sec\theta$ ***B***

10. $2\sqrt{(x-4)^2 + (y-6)^2} = \sqrt{(x+1)^2 + (y-2)^2}$; $4(x^2 - 8x + y^2 - 12y + 52) = x^2 + 2x + y^2 - 4y + 5$
    $3x^2 + 3y^2 - 34x - 44y + 203 = 0$. ***A***

11. $x + y = 3t$, $x + y - z = 3t - 3t + 4 \rightarrow x + y - z = 4$; $d = \left|\frac{1(1)-4}{\sqrt{3}}\right| = \sqrt{3}$. **C**

12. $\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y-4)^2}$; $2x + y = 5$;
    $\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-5)^2 + (y-2)^2}$; $x = 2$; $y = 1$; $C(2, 1)$; $r = \sqrt{3^2 + 1} = \sqrt{10}$.
    Area = 10π. ***B***

13. $M_{pt} = (2, 2)$; $d = \sqrt{0+1} = 1$. ***A***

14. $A\Delta = \frac{1}{2}\begin{vmatrix} -1 & 2 \\ 3 & 4 \\ 5 & 2 \\ -1 & 2 \end{vmatrix} = \frac{1}{2}|(12 - 24)| = 12$. $10\pi - 12$. ***E***

15. $(3 - 2)\boldsymbol{i} + (0 + 2)\boldsymbol{j} + (1 - 1)\boldsymbol{k}$; $(3 + 2)\boldsymbol{i} + (0 + 4)\boldsymbol{j} + (1 - 2)\boldsymbol{k} \rightarrow \boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ 5 & 4 & -1 \end{vmatrix} = \boldsymbol{i}(-2) - \boldsymbol{j}(-1) + \boldsymbol{k}(-6)$
$\boldsymbol{n} = -2\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k}$. $-2(x - 2) + y - 6(z - 1) = 0 \rightarrow -2x + y - 6z = -12$. ***C***

16. $y = 40\left(\frac{x}{4}\right) - 16\left(\frac{x^2}{16}\right)$; $y = 10x - x^2$; $y = x(10 - x)$. ***C***

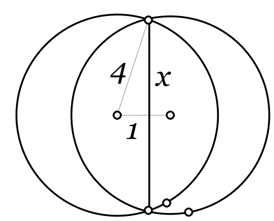
17. $m_1 = 3$; $m_2 = -1$; $\tan\theta = \left|\frac{3-(-1)}{1+(3)(-1)}\right|$; $\theta = \arctan(2)$. ***E***

18. $2\sqrt{(x - 1)^2 + y^2} = \sqrt{(x + 2)^2 + y^2} \rightarrow 3x^2 + 3y^2 - 12x = 0$; $(x - 2)^2 + y^2 = 4$ Circle: $e = 0$ ***A***

19. Such a line has the form $y - 7 = m(x + 6) \rightarrow mx + 6m - y + 7 = 0$. Use the distance from point to line formula to obtain $\frac{6m+7}{\sqrt{m^2+1}} = 2 \rightarrow 36m^2 + 84m + 49 = 4m^2 + 4 \rightarrow 32m^2 + 84m + 45 = (8m + 15)(4m + 3)$. The slopes are $-15/8$ and $-3/4$. Therefore, the answer is $-15 - 3 = -18$. ***D***

20.

$x = \sqrt{15}$, $m\angle = \tan^{-1}(\sqrt{15}) = 75^\circ$
m Arc $= 150^\circ$ and Area $= \frac{150}{360}(16\pi) = \frac{20\pi}{3}$;
$2\left(\frac{20\pi}{3}\right) = \frac{40\pi}{3}$; Area$\Delta = 2\sqrt{15}$
Area $= \frac{40\pi}{3} - 2\sqrt{15}$. ***B***



21. $m = -\frac{1}{3}$, $y = -\frac{1}{3}x + 1$, (x, y) lies on segment on given line, Substituting: $\frac{1}{2x^2 - x + 3} = f(x, y)$
$f(x, y)$ obtains a maximum when $2x^2 - x + 3$ is a minimum $\rightarrow 2(x^2 - \frac{1}{2}x + \frac{1}{16}) + \frac{11}{8}$, $V(1/4, 11/8)$. $f(x, y) = 8/23$. ***B***

22. $V(2, 3 + 1) = V(2, 4)$. ***C***

23. $d = \sqrt{(x - 1)^2 + y^2} = \sqrt{x^2 - 2x + 1 + 4 - 4x^2} = \sqrt{5 - 2x - 3x^2}$; $V(-1/3, 16/3) \rightarrow 4(1/9) + y^2 = 4$
$y = \pm\frac{4\sqrt{2}}{3}$, $abcd = -\frac{32}{81}$. ***A***

24. $(x - 1)^2 + (y - 3)^2 + (z + 4)^2 = 64$. $r = 8$. ***B***

25. $x^2 + y^2 - 2\sqrt{3}x = 6 \rightarrow C(\sqrt{3}, 0)$ and the point is $(-2\sqrt{3}, 2)$. The distance here is $\sqrt{27 + 4} = \sqrt{31}$.
$D = \sqrt{31} - \sqrt{3}$. ***C***

26. $3(x^2 - 4x + 4) - 8(y^2 - y + 1/4) = 48$; $\frac{(x-2)^2}{16} - \frac{(y-\frac{1}{2})^2}{6} = 1$. $C(2, 1/2)$, $c^2 = 22$. $d = \sqrt{22}$ ***C***

27. A parabola is defined as the set of points equidistant from a point(focus) and a line(directrix). Therefore, since the distance from the parabola to the directrix is 4, the distance from the parabola to the focus is 4. ***B***

28. $m_{AC} = \frac{-1}{11}$, $m\perp = 11$, $y - 5 = 11(x - 3) \rightarrow 11x - y = 28$. ***D***

29. $\text{Midpt}_{AB} = (-1, 3/2)$. $d = \sqrt{(6+1)^2 + (-3-3/2)^2} = \frac{\sqrt{277}}{2}$. ***A***

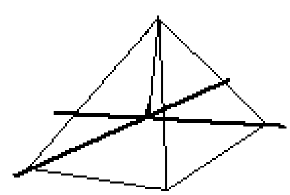
30. $m_{\text{median}} = \frac{-3-3/2}{6+1} = -\frac{9}{14}$, $y + 3 = -\frac{9}{14}(x - 6) \rightarrow 9x + 14y = 12$; Solving for pt. on intersection: $\left(\frac{404}{163}, -\frac{120}{163}\right)$. $a + b + c = 447$. ***C***

Tie-Breakers:

1. DB + DC = AD $\rightarrow \sqrt{(2-0)^2+(y-5)^2}+\sqrt{(0-2)^2+(y-5)^2} = y \rightarrow 2\sqrt{y^2-10y+29} = y \rightarrow$ $3y^2 - 40y + 116 = 0$ has two real solutions but you do not need to find them to evaluate their product. $c/a = 116/3$.

2. Place a vertex of the base at the origin with two of the base edges along the x- and y- axes as shown. We want to get our hands on three vectors. v<2, 0, 0>, u<0, 2, 0>, and w<1, 1, 4>. $v \times w = <0,-8,2>$ and $u \times w = <8,0,-2>$. Remembering that $\cos(\theta) = \frac{a \cdot b}{|a||b|}$, we have that

$$\cos(\theta) = \frac{-4}{\left(\sqrt{68}\right)^2} = -\frac{1}{17}.$$



3. C(4, 5), $a = 5$, $b = 3 \xrightarrow{yields} \frac{(x-4)^2}{9} + \frac{(y\ 5)^2}{25} = 1$.