Proofs Test—SOLUTIONS National MAO Test 2008

1. B—Consider any unbroken 20-hour period and let dscs, dscn, dncs, dncn be the total times when "dog is asleep, cat is asleep", "dog is asleep, cat is not", etc. Claim: all four times are equal. Proof. Break the 20 hour interval into the first 10 hours and second 10 hours and attach digits 1 and 2 to break each of the 4 times into 2 corresponding parts: dscs=dscs1+dscs2, dscn=dscn1+dscn2, etc. The dog's pattern has period 10; so dscs1+dscn1=dscs2+dscn2, etc. The cat's pattern is periodic with period 4. Hence, during the second 10-hour period the cat switches its pattern to the opposite of the first 10-hour period. Hence dscn2=dscs1, dscs2=dscn1, dncs2=dncn1, dncn2=dncs1. Note that the dog sleeps exactly a half of any unbroken 10-hour period. Hence dscs1+dscs2=dscs1+dscn1=5 and the claim is true. It follows that in any 24 hour unbroken interval both animals will be asleep for at least 5 hours. It is easy to arrange the 24-hour period so that the dog sleeps during the last 4 hours.

2. A—The first statement translates to (LF) implies NOT(GE), the second to (W) implies (LF) and the third (T) implies (W), where (LF) abbreviates "loves fish" (GE) abbreviates green eyes, (W) abbreviates "has whiskers" and (T) abbreviates "has a tail". Putting these together yields (T) implies NOT(GE), which is the translation of sentence (a).

3. B—Any handshake will preserve the relative parity of the 3 animals. That is, after any number of moves, the number of Trolls (T's) and the number of Dragons (D's) will either both be even, or they will both be odd. As well, the number of Griffins (G's) will have the opposite parity. At the end of any play of the game, the number of G's cannot be zero, since it's parity always differs from the other two. So there are only G's left at the end. Furthermore, since there are an even number of T's and D's, there must be an odd number of G's at the end of any play of the game. It is easy to see that there are plays of the game that result in any odd number less than 4003 G's being left. For example, to see that 25 Griffins is possible, first have a G and a D shake hands, resulting in 2002 T's and D's and 2001 G's. Then, after 13 T+D handshakes there are 1989 T's and D's, and 2014 G's. Now do 1989 rounds of triple handshakes, first T+G, then D+G, then T+D.

4. B

5. B. Since the conditional is true, the contrapositive is also true. The converse, “If a pentagon is equiangular, then it is regular” is false as in the inverse.

6. A—If n had Property T, the sum of the n subsets would just be the sum of the integers from 1 to $3n$, or $\left(\frac{1+3n}{2}\right)3n$. This in turn would equal nm^2, so $3(3n+1)=2m^2$. This means that 3 must divide m, but then that would force 9 to divide $3(3n+1)$, which is impossible.

7. A—$T_n = n(n+1)$. This must have only 2 as its factors in order to be a power of 2, but this is impossible since neither n or $n+1$ is odd.

8. D— ~ represents multiplication. Since 1001 = 7·11·13, the 7th 11th and 13th letters are the solution. Those letters are G, K and M.

9. B—This is the Pigeonhole Principle.

10. A—Though introduced in 1852 (by Gauss), it was not proven until 1976.

11. C—This is the first step in a mathematical induction proof.

12. B—If there are just two squares we clearly need just one break. Assume that for numbers $2<m<N$ we have already shown that it takes exactly m-1 breaks to split a bar consisting of m squares. Let there be a bar of N squares. Split it into two with m_1 and m_2 squares, respectively. Of course, $m_1 + m_2 = N$. By the induction hypothesis it will take (m_1-1) breaks to split the first bar and (m_2-1) to split the second one. The total will be $1 + (m_1-1)+(m_2-1) = N-1$.

13. D—The flaw occurs in line 7, where the square root of each side is taken (it is not always +).

14. E—8--The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The
three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such
telephone number is 810-642-9753.

15. D—After the first transfer, the first cup contains two ounces of coffee, and the second cup contains two ounces of coffee and four ounces of cream. After the second transfer, the first cup contains 2 + (1/2)(2) = 3 ounces of coffee and (1/2)(4) = 2 ounces of cream. Therefore, the fraction of the liquid in the first cup that is cream is 2/(2 + 3) = 2/5.