Alpha Sequences and Series Solutions

1. $r = \frac{a_2}{a_1} = \frac{\frac{1}{6}}{\frac{1}{3}} = \boxed{\frac{1}{2}}$.

2. $5 + 10 + ... + 160 = \frac{32(5+160)}{2} = \boxed{2640}$.

3. $a_1 = 5$, $r = \frac{2}{5}$, so $S = \frac{5}{1-\frac{2}{5}} = \boxed{\frac{25}{3}}$.

4. $d = a_2 - a_1 = 2x - y - (x + y) = \boxed{x - 2y}$.

5. The sum of the first n odd positive integers is n^2. $66^2 = \boxed{4356}$.

6. The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \boxed{233}, \ldots$.

7. $a_1 = \frac{2}{7}$, $r = 2$, so $S_{10} = \frac{2}{7}(2^{10} - 1) = \boxed{\frac{2046}{7}}$.

8. The first person shakes hands with the 34 others and leaves. The second person shakes hands with the remaining 33 and leaves, and so on. $34 + 33 + 32 + ... + 1 = \frac{34}{2}(1 + 34) = \boxed{595}$.

9. The nth triangular number is the same as the sum of the first n positive integers. $T_{29} = \frac{29(1+29)}{2} = \boxed{435}$.

10. $a_1 = 16$, $d = -2$, and $n = 1 + \frac{-40-16}{-2} = 29$. $S_{29} = \frac{29(16-40)}{2} = \boxed{-348}$.

11. $a_1 = -40$, $d = -12$, so $a_{100} = -40 + 99 \cdot (-12) = \boxed{-1228}$.

12. The number of rooms R on floor n is given by $R(n) = 5n - 4$.
    $\sum_{n=1}^{25}(5n - 4) = 5 \cdot \frac{25(1+25)}{2} - 25 \cdot 4 = \boxed{1525}$.

13. $a_6 = \frac{1}{6} + \sin(\frac{\pi}{6}) = \frac{1}{6} + \frac{1}{2} = \boxed{\frac{2}{3}}$.

14. The quantity of Cesium-137 during year t is given by $Q(t) = Q_0 \cdot 2^{-\frac{t}{30}}$ where Q_0 is the inital amount.

$$\begin{aligned} 100 &= Q_0 \cdot 2^{-\frac{105}{30}} \\ 100 &= Q_0 \cdot 2^{-\frac{7}{2}} \\ Q_0 &= 100 \cdot 8\sqrt{2} \\ Q_0 &\approx 800 \cdot 1.414 \\ Q_0 &\approx \boxed{1131} \end{aligned}$$

15. Using the facts $i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$ and $i^n = i^{n \mod 4}$, we can group the terms by fours to allow most of the terms to vanish. Terms 1-4, 5-8, ..., 2001-2004 add to zero, so we are left with the final term $i^{2005} = i^{2005 \mod 4} = \boxed{i}$.

16. The arithmetic mean is the sum of all terms divided by the number of terms. $S_n = \frac{n}{2}(a_1 + a_n)$ so the mean is simply the average of the first and last term. $\frac{1}{2}(17 + 83) = \boxed{50}$.

17. $\sum_{n=1}^{k}(n \cdot n!) = (k + 1)! - 1$. $\sum_{n=2}^{1000}(n \cdot n!) = (1001! - 1) - 1 \cdot 1! = \boxed{1001! - 2}$.

18. Thom will win the bet if Thom wins the game. Thom can win the game if triples are rolled on the 2nd, 4th, 6th, 8th, etc. trial (since Tom goes first). The probability of rolling triples on 3 six-sided dice is $P(T) = \frac{6}{6^3} = \frac{1}{36}$. Therefore, the probability that Thom will win is the infinite sum:

$P(T)^C \cdot P(T) + (P(T)^C)^3 \cdot P(T) + (P(T)^C)^5 \cdot P(T) + \ldots = P(T) \cdot \frac{P(T)^C}{1-(P(T)^C)^2}$

$P(T)^C = 1 - P(T) = \frac{35}{36}$, so we have $\frac{1}{36} \cdot \frac{\frac{35}{36}}{1-\frac{35^2}{36^2}} = \frac{35}{36^2(1-\frac{35^2}{36^2})} = \frac{35}{36^2-35^2} = \frac{35}{(36-35)(36+35)} = \boxed{\frac{35}{71}}$.

19. $P_1 = 1, P_2 = 2, P_3 = 2 \cdot 2 + 1 = 5, P_4 = 2 \cdot 5 + 2 = 12, P_5 = 2 \cdot 12 + 5 = \boxed{29}$.

20. The tennis ball travels upward a total distance of $\frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \ldots = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{5}{2}$ meters. The balls travels downward a total of $5 + \frac{5}{3} + \frac{5}{9} + \ldots = 5 + \frac{5}{2}$ meters. The total vertical distance is $\frac{5}{2} + 5 + \frac{5}{2} = \boxed{10 \text{ m}}$.

21. The product represents the product of the odd numbers from 1 to $2n-1$, which is equal to $(2n-1)!$ with all the even numbers divided out. Thus, we have $\frac{(2n-1)!}{2\cdot4\cdot6\cdots(2n-2)} = \frac{(2n-1)!}{2^n(1\cdot2\cdot3\cdots(n-1))} = \frac{(2n-1)!}{2^{n-1}\cdot(n-1)!}$.

Multiply by $2n$ on the top and bottom to obtain $\boxed{\frac{(2n)!}{2^n \cdot n!}}$

22. We are seeking the least term greater than $\frac{7}{5} = 1.4$. The first few terms of this sequence are $M_1 = 1048576 = 2^{20}, M_2 = 2^{10}, M_3 = 2^5, M_4 = 2^{\frac{5}{2}} = 4\sqrt{2}, M_5 = 2^{\frac{5}{4}} = 2 \cdot \sqrt[4]{2}, M_6 = 2^{\frac{5}{8}} > 2^{\frac{1}{2}} > 1.4$. The next term, $M_7 = 2^{\frac{5}{16}}$, is a fair amount less than $\sqrt{2}$ and thus less than 1.4. So the value of n we are looking for is $\boxed{6}$.

23. $\prod_{j=3}^{1000}(j+1) = 4 \cdot 5 \cdot 6 \cdots 1001 = \frac{1001!}{3!} = \boxed{\frac{1001!}{6}}$.

24. The third entry in the nth row of Pascal's Triangle is ${}_nC_2 = \frac{n(n-1)}{2}$.

$\sum_{n=2}^{60} \frac{n(n-1)}{2} = \frac{1}{2}\sum_{n=2}^{60}(n^2-n) = \frac{1}{2}\sum_{n=2}^{60} n^2 - \frac{1}{2}\sum_{n=2}^{60} n = \frac{60(61)(121)}{12} - \frac{1}{2} - \left(\frac{60(61)}{4} - \frac{1}{2}\right) = 36905 - 915 = \boxed{35990}$.

25. Using the identities $\sin(x) = \cos(90° - x)$ and $\sin(x) = \sin(180° - x)$ we can rewrite the sum as follows:

$$\begin{aligned}
\sum_{k=1}^{179} \sin^2(k°) &= \\
\sum_{k=1}^{44}(\sin^2(k°)) + \sin^2(45°) + \sum_{k=46}^{89}(\sin^2(k°)) + \sin^2(90°) & \\
+ \sum_{k=91}^{134}(\sin^2(k°)) + \sin^2(135°) + \sum_{k=136}^{179}(\sin^2(k°)) &= \\
\sum_{k=1}^{44}(\sin^2(k°) + \cos^2(k°)) + \sin^2(45°) + \sin^2(90°) & \\
+ \sin^2(135°) + \sum_{k=1}^{44}(\sin^2(k°) + \cos^2(k°)) &= \\
44 + \frac{1}{2} + 1 + \frac{1}{2} + 44 &= \boxed{90}
\end{aligned}$$

26. Let one side of the triangle measure 1 unit. Then we have remaining sides r and r^2 (for $r > 0$ of course). Using the triangle inequality, we know that the largest side can measure no more than the sum of the two smaller side lengths. If $0 < r < 1$ then 1 is the largest side length; otherwise, r^2 is the largest side length. We can then write two inequalities:

$$\begin{aligned} r^2 &< 1+r & (1)\\ 1 &< r^2+r & (2)\\ &\Downarrow \\ r^2-r-1 &< 0 & (3)\\ r^2+r-1 &> 0 & (4)\\ &\Downarrow \\ \left(r-\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(r-\left(\frac{1-\sqrt{5}}{2}\right)\right) &< 0 & (5)\\ \left(r-\left(\frac{-1+\sqrt{5}}{2}\right)\right)\left(r-\left(\frac{-1-\sqrt{5}}{2}\right)\right) &> 0 & (6) \end{aligned}$$

Both $\frac{1-\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$ are less than zero, so we need not consider these values when testing values between the critical numbers.

In (5) we first test a value of r such that $r > \frac{1+\sqrt{5}}{2}$; clearly, any such value does not satisfy the inequality. Next, we try a value of r such that $0 < r < \frac{1+\sqrt{5}}{2}$; this makes the inequality true so the solution for (5) is $0 < r < \frac{1+\sqrt{5}}{2}$.

In (6) we first test a value of r such that $0 < r < \frac{-1+\sqrt{5}}{2}$; clearly, any such value does not satisfy the inequality. Next, we try a value of r such that $r > \frac{-1+\sqrt{5}}{2}$; this makes the inequality true so the solution for (6) is $r > \frac{-1+\sqrt{5}}{2}$.

The intersection of these two solution sets describes the allowed values of r: $\frac{-1+\sqrt{5}}{2} < r < \frac{1+\sqrt{5}}{2}$. $A+B$ is then $\frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \boxed{\sqrt{5}}$.

27. Compute the partial fraction decomposition of $\frac{3}{x^3-x}$:

$$\begin{aligned} \frac{3}{x^3-x} &= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}\\ 3 &= A(x^2-1)+B(x^2+x)+C(x^2-x)\\ 3 &= (A+B+C)x^2+(B-C)x-A \end{aligned}$$

Thus $A=-3$, $B+C=3$, and $B=C=\frac{3}{2}$.

$$\begin{aligned} \sum_{x=2}^{\infty}\frac{3}{x^3-x} &= \sum_{x=2}^{\infty}\left(\frac{3}{2(x-1)}+\frac{3}{2(x+1)}-\frac{3}{x}\right)\\ &= \sum_{x=2}^{\infty}3\cdot\left[\frac{1}{2(x-1)}+\frac{1}{2(x+1)}-\frac{1}{x}\right]\\ &= 3\cdot\left[\left(\frac{1}{2}+\frac{1}{6}-\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{8}-\frac{1}{3}\right)+\left(\frac{1}{6}+\frac{1}{10}-\frac{1}{4}\right)+\ldots\right] \end{aligned}$$

Line up the terms of each decomposed factor separately:

$3(\frac{1}{2} \quad +\frac{1}{4} \quad +\frac{1}{6} \quad +\frac{1}{8} \quad +\frac{1}{10} \quad +\ldots)$

$\qquad\qquad 3(\frac{1}{6} \quad +\frac{1}{8} \quad +\frac{1}{10} \quad +\ldots)$

$-3(\frac{1}{2} \quad +\frac{1}{3} \quad +\frac{1}{4} \quad +\frac{1}{5} \quad +\frac{1}{6} \quad +\ldots)$

Notice that the terms of the top two sums can be easily combined starting with $\frac{1}{6}$. We can simplify this to $3\left[\frac{1}{2}+\frac{1}{4}+\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\right)-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\right)\right]=\boxed{\frac{3}{4}}$.

28. $[k;\overline{k}] = k + \frac{1}{k+\frac{1}{k+\ldots}}$ which is not much less than $k+\frac{1}{k}$. $\sum_{k=1000}^{2000}\left(k+\frac{1}{k}\right) = \frac{1}{2}(1001)(1000+2000) + \sum_{k=1000}^{2000}\frac{1}{k} < 1501500 + 1001\cdot\frac{1}{1000} \approx \boxed{1501501}$.

29. $\sum_{y=1}^{6}\sum_{x=1}^{6}(x+y)^2 = \sum_{y=1}^{6}\sum_{x=1}^{6}(x^2+2xy+y^2) =$
$\sum_{y=1}^{6}\sum_{x=1}^{6}x^2 + \sum_{y=1}^{6}\sum_{x=1}^{6}y^2 + 2\sum_{y=1}^{6}\sum_{x=1}^{6}xy =$
$2\cdot 6\cdot\frac{6\cdot 7\cdot 13}{6} + 2\cdot\frac{1}{4}\cdot(6\cdot 7)(6\cdot 7) = \boxed{1974}$

30. Drop an altitude from angle C intersecting with side AB at point D. Now draw a line segment perpendicular to BC starting from D ending at point E on side BC. Draw another line segment perpendicular to AB ending at point F on AB. These new line segments form the path that the ant walks along. All right triangles formed by the new line segments are similar to each other.

Extending the pattern of line segments indefinitely, we note that the lengths of the segments form two geometric series, one with terms $\overline{AC}, \overline{DE}, \ldots$ and another with terms $\overline{CD}, \overline{EF}, \ldots$. $\overline{AC} = 6\tan(15^\circ)$, $\overline{CD} = 6\sin(15^\circ)$, $\overline{DE} = 6\sin(15^\circ)\cos(15^\circ)$, and $\overline{EF} = 6\sin(15^\circ)\cos^2(15^\circ)$. This makes the common ratio of the both series $r = \cos^2(15^\circ)$. The sum of these two infinite geometric series is $S = \frac{\overline{AC}+\overline{CD}}{1-r} = \frac{6\tan(15^\circ)+6\sin(15^\circ)}{1-\cos^2(15^\circ)} = 6\cdot\frac{\tan(15^\circ)+\sin(15^\circ)}{\sin^2(15^\circ)} = \frac{6}{\sin(15^\circ)\cos(15^\circ)} + \frac{6}{\sin(15^\circ)}$.

By the difference formula, $\sin(15^\circ) = \sin(45^\circ - 30^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$, so $S = \frac{6}{\frac{1}{2}\sin(30^\circ)} + \frac{24}{\sqrt{6}-\sqrt{2}} = 24 + \frac{24\cdot(\sqrt{6}+\sqrt{2})}{4} = \boxed{24+6\sqrt{6}+6\sqrt{2}}$.