Limits and Derivatives
Mu Alpha Theta State Convention 2004
SOLUTIONS

1. D
2. B
3. E
4. B
5. C
6. B
7. D
8. C
9. A
10. C
11. D
12. A
13. B
14. C
15. C
16. C
17. B
18. C
19. A
20. E
21. B
22. D
23. E
24. A
25. A
26. B
27. A
28. A
29. B
30. C

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1. (D)

$$x(t)=8t^4-2t^2+9$$
$$v(t)=32t^3-4t$$
$$a(t)=96t^2-4$$
$$a'(t)=192t \qquad @\,t=1 \qquad a'(1)=192$$

2. (B)

$$\lim_{x\to 0}\cot(7x)\sin(4x)=\lim_{x\to 0}\frac{\sin(4x)}{\sin(7x)}\cdot\cos(7x)$$
$$=\left(\lim_{x\to 0}\frac{\sin(4x)}{4x}\lim_{x\to 0}\frac{7x}{\sin(7x)}\lim_{x\to 0}\cos(7x)\right)\frac{4}{7}=\frac{4}{7}$$

3. (E) $$\lim_{x\to\infty}\frac{a+bx^3}{c-dx^3}=\lim_{x\to\infty}\frac{\frac{a}{x^3}+b}{\frac{c}{x^3}-d}=\lim_{x\to\infty}\frac{-b}{d}=\frac{-b}{d}$$

4. (B) If $f(x)$ is even, then $f'(x)$ is odd. Therefore, $f'(-x)=-f'(x)$

5. (C) The probability of not getting the #1 ball is $\frac{n-1}{n}$, which happens for n independent events, which we must take the limit to infinity.

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^{-n}=e^{-1}\approx 0.368$$

6. (B)

$$f(x)=\frac{1}{4}x^4-3x^3+2x^2 \qquad f'(x)=x^3-9x^2+4x$$
$$f''(x)=3x^2-18x+4=0 \qquad \Rightarrow x=\frac{9\pm\sqrt{69}}{3}$$

By checking $f''(x)$ at each of these points, we find that signs change, so they are both inflection points.

7. (D) $$\frac{d}{dx}\int_1^{x^3}t^2\sin t\,dt=(dt)(x^3)^2\sin(x^3)=(3x^2)x^6\sin(x^3)=3x^8\sin(x^3)$$

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8. (C)

$$3x^2y^2 + 5y^3x = 6 \qquad \frac{dy}{dx} = \frac{-\frac{\partial}{\partial y}(3x^2y^2 + 5y^3x)}{\frac{\partial}{\partial x}(3x^2y^2 + 5y^3x)}$$

$$= \frac{-6x^2y - 15y^2x}{6xy^2 + 5y^3}$$

$$@\ (2,2) \qquad = \frac{-21}{11}$$

9. (A)

$$y = \frac{1}{\sqrt{x^3+1}} \qquad y' = \frac{-3x^2}{2\sqrt{(x^3+1)^3}} \qquad y'' = \frac{15x^4 - 12x}{4\sqrt{(x^3+1)^5}}$$

$$y''(2) = \frac{15\cdot 16 - 12\cdot 2}{4\sqrt{(8+1)^5}} = \frac{2}{9}$$

10. (C)

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = \frac{\sin(\pi) - \sin(0)}{\pi - 0} = 0$$

$$f'(c) = \cos(c) = 0 \quad \therefore c = \frac{\pi}{2}$$

11. (D) Total Perimeter = 12 ft.

$$4x + 2\pi r = 12 = P \qquad \Rightarrow r = \frac{6-2x}{\pi}$$

$$A = x^2 + \pi r^2 = x^2 + \pi\left(\frac{6-2x}{\pi}\right)^2 = \frac{1}{\pi}\left((\pi+4)x^2 - 24x + 36\right)$$

$$\frac{dA}{dx} = \frac{2}{\pi}\left((\pi+4)x - 12\right) = 0 \qquad \therefore x = \frac{12}{\pi+4} \approx 1.68\, ft$$

$$r \approx 0.84\, ft$$

$$Square\ needs\ 4x = 6.72\, ft \qquad Circle\ needs\ 2\pi r = 5.28\, ft$$

12. (A)

$$f(g(-2)) = -2 \qquad f'(-1) = 10(-1)^4 + 3(-1)^2 + 1 = 14$$

$$-2 = 2x^5 + x^3 + 1 \qquad g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{14}$$

$$x = -1 = g(-2)$$

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13. (B) $\frac{d}{dx}[\arctan x] = \frac{x'}{x^2+1}$ $\frac{d}{dx}\left[\arctan \frac{3}{5}x\right] = \frac{15}{9x^2+25}$

14. (C) I. Differentiable everywhere
II. Not differentiable at $x=0$; it's a vertical line at that point
III. Differentiable everywhere except $x=\frac{1}{2}$ since isn't smooth at that point
IV. Not differentiable at $x=0$ because it is not continuous at that point

Not differentiable at $x=0$ for II and IV

15. (C) $\lim_{m\to 3}\frac{m^4-18m^3+116m^2-318m+315}{m^2-6m+9} = \lim_{m\to 3}(m-7)(m-5) = 8$

16. (C) $\ln\left[\lim_{x\to 1} x^{\frac{1}{x-1}}\right] = \lim_{x\to 1}\left[\frac{1}{x-1}\cdot\ln(x)\right] = \lim_{x\to 1}\frac{\ln(x)}{x-1} = \lim_{x\to 1}\frac{1}{x} = 1$

$\therefore \lim_{x\to 1} x^{\frac{1}{x-1}} = e^1 = e$

17. (B) $\lim_{x\to\infty}\frac{\sqrt{4x^2+2}}{3x+1} = \lim_{x\to\infty}\frac{\frac{\sqrt{4x^2+2}}{x}}{\frac{3x+1}{x}} = \lim_{x\to\infty}\frac{\sqrt{\frac{4x^2}{x^2}+\frac{2}{x^2}}}{\frac{3x}{x}+\frac{1}{x}}$

$= \lim_{x\to\infty}\frac{\sqrt{4+0}}{3+0} = \lim_{x\to\infty}\frac{2}{3} = \frac{2}{3}$

18. (C) $f(x) = x^3 - x + 5$ $f'(x) = 3x^2 - 1 = 0$

$x = \pm\frac{1}{\sqrt{3}}$

$f'(-1) = 3(+)$ $f'(0) = -1(-)$ $f'(1) = 3(+)$

X is increasing on $(-\infty, \frac{-1}{\sqrt{3}}]$ U $[\frac{1}{\sqrt{3}}, \infty)$

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19. (A)

$$a(n) = \sum_{i=1}^{n}\left(3 + \frac{2i}{n}\right)^2\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3 + \frac{2i}{n}\right)^2\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9}{n} + \frac{12i}{n^2} + \frac{4i^2}{n^3}\right) =$$

$$\lim_{n\to\infty}\frac{9}{n}n + \frac{12}{n^2}\frac{n(n+1)}{2} + \frac{4}{n^3}\frac{(2n+1)(n+1)(n)}{6} =$$

$$\lim_{n\to\infty}9 + \frac{12n^2 + 12n}{2n^2} + \frac{8n^3 + 12n^2 + 4n}{6n^3} = 9 + \frac{12}{2} + \frac{8}{6} = \frac{53}{3}$$

20. (E) sin(x) and cos(x) are equivalent to their n mod 4 derivatives. Also, $\sum_{i=1}^{2004} f^{(i)}(\pi) = 0$ since every set of 4 cancels out (0+1+0+-1=0).

Therefore, $\sum_{i=0}^{2004} f^{(i)}(\pi) = f(\pi) = \cos(\pi) - \sin(\pi) = -1$

21. (B) $f(x) = \dfrac{2x^3 + 3x^2 - 32x + 15}{(x-3)}$ has a removable discontinuity at 3

$$\therefore f(3) = \frac{2(x-3)(x+5)(x-.5)}{(x-3)} = 2(x+5)(x-.5) = 2(8)(2.5) = 40$$

22. (D)

$$f(x+\Delta x) = f(x) + f'(x)\Delta x = \frac{1}{\sqrt[3]{2x+3}} + \frac{-2}{3\sqrt[3]{(2x+3)^4}}\Delta x$$

$$= \frac{1}{\sqrt[3]{2(12)+3}} + \frac{-2}{3\sqrt[3]{(2(12)+3)^4}}4 \approx .300$$

23. (E) Each derivative multiplies the function by $b\ln a$.

$f(x) = a^{bx}$ $\quad f'(x) = (b\ln a)a^{bx}$

$f''(x) = (b\ln a)^2 a^{bx}$ $\quad f'''(x) = (b\ln a)^3 a^{bx}$

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24. (A) The length of the other side of the triangle is $\sqrt{25^2-15^2}=20$
Using the Pythagorean theorem and total differentiation, we get

$$d\left[x^2+y^2=z^2\right]=x\frac{d}{dx}+y\frac{d}{dy}=z\frac{d}{dz}$$

$$\frac{d}{dz}=0,\frac{d}{dy}=-1.5, x=20, y=15, z=25$$

$$20(\frac{d}{dx})+15(-1.5)=(25)(0) \qquad \Rightarrow \frac{d}{dx}=1.125\approx 1.13$$

25. (A)

$$\lim_{x\to 4}\frac{d}{dx}\left[(x-4)^2 g(x)\right]=\lim_{x\to 4}\frac{d}{dx}\left[x^3-4x^2+2x-1\right]=$$

$$\lim_{x\to 4}3x^2-8x+2=18$$

26. (B)

$$d=\sqrt{(x-1)^2+(y-0)^2}=\sqrt{x^2-2x+1+y^2}=\sqrt{x^2-2x+1+\left(3-5x\right)^2}$$

$$=\sqrt{26x^2-32x+10}$$

$$\frac{dd}{dx}=\frac{\sqrt{2}\left(13x-8\right)}{\sqrt{13x^2-16x+5}}=0 \qquad x=\frac{8}{13}$$

$$y=3-5\left(\frac{8}{13}\right)^2=\frac{187}{169} \qquad \left(\frac{8}{13},\frac{187}{169}\right)$$

27. (A)

$$\sin(z)=\lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-1)^n(z)^{2n+1}}{(2n+1)!} \qquad \frac{\sin(\sqrt{z})}{\sqrt{z}}=\lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-1)^n(z)^n}{(2n+1)!}$$

$$\frac{\sin(z)}{z}=\lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-1)^n(z)^{2n}}{(2n+1)!} \qquad \frac{\sin(\sqrt{2})}{\sqrt{2}}=\lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-1)^n(2)^n}{(2n+1)!}$$

28. (A)

$$P(x)=R(x)-C(x)=x^2-.89x-8000$$

$$\left.\frac{dP(x)}{dx}\right|_{x=1000}=2(1000)-.89=1999.11$$

29. (B)

$$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}=\lim_{x\to 4}\frac{\frac{d}{dx}\left[\sqrt{x+5}-3\right]}{\frac{d}{dx}\left[x-4\right]}=\lim_{x\to 4}\frac{1}{2\sqrt{x+5}}=\frac{1}{2\sqrt{9}}=\frac{1}{6}$$

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30. (C) $\lim_{t\to\infty} 8000 - 4300e^{-0.178t} = 8000 - 0 = 8000$

$$8000 \cdot \frac{3}{4} = 8000 - 4300e^{-0.178t}$$

$$6000 = 8000 - 4300e^{-0.178t}$$

$$\frac{20}{43} = e^{-0.178t} \qquad \therefore t \approx 4.30$$

Since 4.3 years have passed, it occurs in the 5^{th} year, which is 2008.