Differential Equations Solutions
2004 FAMAT State Convention

1. Since the highest order derivative is two, and the differential equation is in the form $y''+f(x)y'+g(x)y=h(x)$, it is second order and linear. **C**

2. Separating variables gives $y'=-\frac{y}{x}$. $\frac{dy}{y}=-\frac{1}{x}$. $\ln|y|=-\ln|x|+C=\ln\left|\frac{C}{x}\right|$. $xy=C$. This is a family of hyperbolas. **C**

3. $\frac{\partial}{\partial y}\left[e^x y^2\right]=2e^x y=\frac{\partial}{\partial x}\left[2e^x y\right]$, therefore $e^x y^2\,dx+2e^x y\,dy=0$ is exact. **B**

4. If $y(t)$ is the number of grams of Rubidium-83 the remains after t days, then $\frac{dy}{dt}=ky$. $y=Ce^{kt}$. Since $y(0)=10$, $C=10$. Since $y(86)=5$, $5=10e^{kt}$ and $k=-\frac{\ln 2}{86}$. $1=10e^{-\frac{\ln 2}{86}t}$. $-\ln 10=\frac{-\ln 2}{86}t$. $t=\frac{86\ln 10}{\ln 2}\approx 285.686\approx 286$. **E**

5. $xy'-y=y^2\sin x$ is a Bernoulli equation, and we use the substitution $v=y^{1-2}$. $v'=-\frac{y'}{y^2}$. Therefore $y=\frac{1}{v}$ and $y'=-y^2v'=-\frac{v'}{v^2}$. Substituting into the equation yields $-\frac{xv'}{v^2}-\frac{1}{v}=\frac{1}{v^2}\sin x$. $xv'+v=-\sin x$. Since $\frac{d}{dx}[xv]=xv'+v$, integrating both sides with respect to x gives $xv=\cos x+C$. Since $v=\frac{1}{y}$, $\frac{x}{y}=\cos x+C$ and $y=\frac{x}{\cos x+C}$. **A**

6. $\frac{y'}{y}=\ln x$. $\ln|y|=x\ln x-x+C$. $y=e^{x\ln x-x+C}=\frac{Cx^x}{e^x}$. $e=\frac{C}{e}$. $C=e^2$. $y(2)=\frac{e^2 2^2}{e^2}=4$. **A**

7. Since $(tx)^2+(ty)^2=t^2x^2+t^2y^2=t^2\left(x^2+y^2\right)$ and $(tx)(ty)=t^2(xy)$, $\left(x^2+y^2\right)dx+xy\,dy=0$ is a homogeneous differential equation and can be made separable by the substitution $y=xv$. $dy=xdv+vdx$. Substituting into the equation yields $\left(x^2+x^2v^2\right)dx+x^2v(xdv+vdx)=0=x^3vdv+\left(x^2+2x^2v^2\right)dx$. Separating variables gives $\frac{vdv}{1+2v^2}=-\frac{x^2}{x^3}dx$. Integrating gives $\ln\left|1+2v^2\right|=\ln\left|\frac{C}{x^4}\right|$. $1+2v^2=\frac{C}{x^4}$. $v^2=\frac{C}{x^4}-\frac{1}{2}$. Since $v=\frac{y}{x}$, $\frac{y^2}{x^2}=\frac{C}{x^4}-\frac{1}{2}$ and $y^2=\frac{C}{x^2}-\frac{x^2}{2}$. **B**

8. Since $F=ma$ and $a=\frac{dv}{dt}$, $3v'=f(t)=\frac{1}{t+1}$. The force due to friction is $-kv$ where k is a positive constant. Therefore the velocity is the solution to $3v'=\frac{1}{t+1}-kv$. **C**

9. $\frac{dy}{y^2}=x^3dx$. $-\frac{1}{y}=\frac{x^4}{4}+C$. $C=-1-0=-1$. $y=\frac{4}{4-x^4}$. $\frac{4}{4-2^4}=\frac{4}{-12}=-\frac{1}{3}$. **B**

10. Notice that $\frac{\partial}{\partial y}\left[y^2e^x+xy^2e^x+3x^2\right]=2ye^x+2xye^x=\frac{\partial}{\partial x}\left[2xye^x\right]$, hence the differential equation is exact. Integrating $2xye^x$ with respect to y yields $xy^2e^x+g(x)$. Differentiating with respect to x yields

$y^2e^x + xy^2e^x + g'(x) = y^2e^x + xy^2e^x + 3x^2$. Hence $g'(x) = 3x^2$ and $g(x) = x^3 + C$. Therefore the general solution is $xy^2e^x + x^3 = C$ or $y^2 = \left(\frac{C}{x} - x^2\right)e^{-x}$, and $y^2 = \left(\frac{7}{x} - x^2\right)e^{-x}$ is a solution. **C**

11. Since $y(1) = 0$, $y'(1) = 1 + 0 = 1$. $y(1.1) \approx 1(1.1 - 1) + 0 = .1$. $y'(1.1) \approx 1.1 + .1 = 1.2$. $y(1.2) \approx 1.2(1.2 - 1.1) + .1 = .12 + .1 = .22$. **B**

12. Let $T(t)$ be the temperature of the water at time t. Therefore $T(0) = 212$, $T(2) = 150$, and $\frac{dT}{dt} = k(T - 70)$ by Newton's Law of Cooling. $\frac{dT}{T-70} = k$. $\ln|T - 70| = kt + C$. $C = \ln 142$. $k = \frac{\ln\frac{40}{71}}{2}$. $t = \frac{\ln 20 - \ln 142}{\frac{\ln\frac{40}{71}}{2}} \approx 6.83197 \approx 6\frac{50}{60}$. 6 minutes 50 seconds. **C**

13. Notice that $\frac{d}{dx}\left[y^2e^{x^2}\right] = 2ye^{x^2}y' + 2xy^2e^{x^2}$. Hence $y^2e^{x^2} = \frac{x^2}{2} + 3x + C$ and $y^2 = \left(\frac{x^2}{2} + 3x + C\right)e^{-x^2}$. **A**

14. $a(t) = -32$, $v(t) = -32t + C_1$, and $s(t) = -16t^2 + C_1t + C_2$. Since $s(0) = 0$, $C_2 = 0$. Since $s(2.5) = 0$, $-100 + 2.5C_1 = 0$ and $C_1 = 40$. The maximum will occur when the velocity is 0, hence $t = \frac{40}{32} = 1.25$. $s(1.25) = -16 \cdot 1.25^2 + 40 \cdot 1.25 = 25$. **A**

15. Since $\frac{\frac{\partial}{\partial y}\left[x^3(2y-3)\right] - \frac{\partial}{\partial x}\left[x^2\right]}{x^2} = \frac{2x^3 - 2x}{x^2} = 2x - \frac{2}{x}$ is a function of x alone, $e^{\int\left(2x - \frac{2}{x}\right)dx} = e^{x^2 - 2\ln x} = \frac{e^{x^2}}{x^2}$ is an integrating factor. **D**

16. Let $P(t)$ be the amount owed after t months and k the monthly payment. The monthly interest rate is $\frac{.06}{12} = .005$. $P(0) = 10{,}000$ and $P(48) = 0$. The differential equation is $\frac{dP}{dt} = .005P - k$. $\frac{dP}{.005P - k} = dt$. $\ln|.005P - k| = .005t + C$. $.005P - k = Ce^{.005t}$. $P = Ce^{.005t} + 200k$. $10000 = C + 200k$ and $C = \frac{-200k}{e^{.24}}$. $10000 = 200k\left(1 - e^{-.24}\right)$. $k = \frac{50}{1 - e^{-.24}} \approx 234.33$. **B**

17. The differential equation is not linear. The degree of the differential equation is the degree of the highest order derivative, which is 1. The order is 4. If $y = \cos x$, then $\left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^4y}{dx^4}\right) = (-\sin x)^2 + \cos^2 x = 1$, hence $y = \cos x$ is a solution. $y = a(x) + b(x)$ is not a solution because the differential equation is not linear. Notice that $y = \cos x$ and $y = x$ are both solutions, but $y = \cos x + x$ is not. **B**

18. $y = -\frac{\cos(2x)}{2} + \frac{x^2}{2} + C$. Since $y(0) = 0$, $C = \frac{1}{2}$. $y\left(\frac{\pi}{2}\right) = -\frac{-1}{2} + \frac{\pi^2}{8} + \frac{1}{2} = \frac{\pi^2 + 8}{8}$. **D**

19. If the differential equation is exact, then $M_y(x,y) = N_x(x,y)$, hence $\left[M_y(x,y)\right]^3 - \left[N_x(x,y)\right]^3 = 0$. **A**

20. $y' - \frac{1}{x^2}y = -\frac{1}{x^2}$. The integrating factor is $e^{\int -\frac{1}{x^2}dx} = e^{1/x}$. $e^{1/x}y' - \frac{e^{1/x}}{x^2}y = -\frac{e^{1/x}}{x^2}$. $e^{1/x}y = e^{1/x} + C$ and $y = Ce^{-1/x} + 1$. **E**

21. $\frac{f'(x)}{f(x)} = 2$. $\ln|f(x)| = 2x + C$. $f(x) = Ce^{2x}$. Since $f(0) = 1$, $C = 1$. $f(1) = e^2$. **E**

22. $\frac{3}{2}$ pounds of salt per minute flow into the tank per minute and $\frac{3}{50}y$ pounds of salt per minute flow out. The differential equation is $y' = \frac{3}{2}t - \frac{3}{50}y$. **D**

23. The differential equation is $\frac{dy}{dt} = ky(423 - y)$ where $y(t)$ is the number of supporters after t weeks and k is a constant. Separating variables yields $\frac{dy}{y(423-y)} = kdt$. By partial fraction decomposition, $\frac{1}{y(423-y)} = \frac{1}{423}\left(\frac{1}{y} + \frac{1}{423-y}\right)$, hence $\left(\frac{1}{y} + \frac{1}{423-y}\right)dy = kdt$. Integrating gives $\ln|y| - \ln|423 - y| = kt + C$. $\frac{y}{423-y} = Ce^{kt}$. $y = \frac{423Ce^{kt}}{1+Ce^{kt}}$. Since $y(0) = 1$, $1 = \frac{423C}{1+C}$ and hence $C = \frac{1}{422}$. Since $y(1) = 25$ and $\frac{y}{423-y} = \frac{e^{kt}}{422}$, $k = \ln\frac{5275}{199}$. $y(2) = \frac{131875}{499} \approx 264$. **C**

24. If $T(Q)$ is the profit function, $T(Q) = R(Q) - C(Q)$ and hence profit is maximized when $T'(Q) = 0$, or when $MR(Q) = MC(Q)$. $R(Q) = 15000Q - 1550Q^2$ and hence $MR(Q) = 15000 - 3100Q$. $15000 - 3100Q = 6000 - 1600Q$ and $Q = 6$. $C(Q) = 6000Q - 800Q^2 + C$, and $C = 2000$ because the fixed cost is \$2000. $T(6) = R(6) - C(6) = 34200 - 9200 = 25000$. **B**

25. All four statements are general properties of logistic equations. **D**

26. Notice that $\frac{\partial}{\partial y}[2x\sin y] = 2x\cos y = \frac{\partial}{\partial x}\left[x^2\cos y + 2y\right]$ hence the equation is exact. $\int 2x\sin y\,dx = x^2\sin y + g(y)$, and $x^2\cos y + g'(y) = x^2\cos y + 2y$. $g(y) = y^2 + C$ and $x^2\sin y + y^2 = C$. **D**

27. Using integration by parts, $\frac{dy}{dt} = -3t\cos t + 3\sin t + C_1$. Since $y'\left(\frac{\pi}{2}\right) = 0$, $C_1 = -3$. $y = -3t\sin t - 6\cos t - 3t + C_2$. Since $y(0) = 2$, $C_2 = 8$. $y = -3t\sin t - 6\cos t - 3t + 8$. **C**

28. There is not enough information given to be able to solve the new differential equation. **D**

29. $x^2y = C$ and $y' = -\frac{2y}{x}$. The orthogonal trajectories are the solutions to $y' = \frac{x}{2y}$. $2y\,dy = x\,dx$. $y^2 = \frac{x^2}{2} + K$. **C**

30. Substituting gives $\frac{g'(x)}{g(x)} = x$. $\ln|g(x)| = \frac{x^2}{2} + C$. $g(x) = Ce^{x^2/2}$. $f(x) = Cxe^{x^2/2}$. $f'(x) = Ce^{x^2/2} + Cx^2e^{x^2/2} = Ce^{x^2/2}(1 + x^2) = g(x)\cdot h(x)$. $h(x) = x^2 + 1$. **B**