Calculus Applications
Mu Alpha Theta State Convention 2004
SOLUTIONS

1. (B)

$$\frac{8}{y}=\frac{20}{x+y} \qquad 8x+8y=20y \qquad 2\frac{dx}{dt}=3\frac{dy}{dt}$$

$$8x=12y \qquad \frac{dy}{dt}=\frac{16}{3}\,ft/s$$

$$2x=3y$$

2. (C)

$$y=\frac{V}{2A}=\frac{\pi\int_0^2 16^2-(x^4)^2\,dx}{2\pi\int_0^2 16-x^4\,dx}=\frac{80}{9}$$

$$x=0 \qquad \left(0,\frac{80}{9}\right)$$

3. (A) $2\int_{-\pi/2}^{\pi/2}\sqrt{(8\sin\theta-8)^2+(8\cos\theta)^2}\,d\theta=64$

4. (A)

$$r(t)=t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}$$

$$r'(t)=1\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}$$

$$r'(2)=1\mathbf{i}+4\mathbf{j}+12\mathbf{k}$$

$$speed=\sqrt{1^2+4^2+12^2}=\sqrt{161}$$

5. (B)

$$v(0)=2000\,ft/s$$

$$a(t)=-32$$

$$v(t)=-32t+2000=0\ @\ t=62.5$$

$$h(t)=-16t^2+2000t$$

$$h(62.5)=62500\,ft$$

6. (D) Using the Shell Method, we calculate this for a semi circle and multiply by 2 (for the 2 halves of a circle):

$$2(2\pi\int_0^2 p(x)h(x)dx)=4\pi\int_0^2(5-x)\sqrt{4-x^2}\,dx==\frac{60\pi^2-32\pi}{3}\approx 163.88$$

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7. (D)

$$3x^3 - x^2 - 10x = -x^2 + 2x$$
$$3x^3 = 12x$$
$$\therefore x = -2, 0, 2$$

From -2 to 0, $f(x) > g(x)$ and from 0 to 2, $g(x) > f(x)$

$$A = \int_{-2}^{0} [f(x) - g(x)]dx + \int_{0}^{2} [g(x) - f(x)]dx$$

$$= \int_{-2}^{0} 3x^3 - 12x\,dx + \int_{0}^{2} -3x^3 + 12x\,dx = \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2}$$

$$= -(12 - 24) + (-12 + 24) = 24$$

8. (C)

$$\int_0^{\pi/4} x \tan x dx = \frac{\frac{\pi}{4} - 0}{2 \cdot 4}\left(f(0) + 2f(\frac{\pi}{16}) + 2f(\frac{\pi}{8}) + 2f(\frac{3\pi}{16}) + f(\frac{\pi}{4}) \right)$$

$$= \frac{\pi}{32}(0 + 2\frac{\pi}{16}\tan\frac{\pi}{16} + 2\frac{\pi}{8}\tan\frac{\pi}{8} + 2\frac{3\pi}{16}\tan\frac{3\pi}{16} + \frac{\pi}{4}\tan\frac{\pi}{4}) \approx 0.194$$

9. (D)

$$80000 = 200000e^{5k}$$
$$.4 = e^{5k}$$
$$\frac{\ln(.4)}{5} = k \approx -0.183258$$

Thus, after 3 more months (t = 8), one can expect his savings to amount to

$$y \approx 200000e^{-0.183258(8)} \approx 46{,}100$$

10. (B)

$$V(t) = \frac{4\pi}{3}(1 + 3t)^3 = \frac{4\pi}{3}r^3 \qquad r = 1 + 3t \qquad \frac{dr}{dt} = 3$$

$$t = 24, r = 73$$

$$S(t) = 4\pi r^2 \quad \Rightarrow \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi(73)(3) \approx 5504.07$$

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

11. (C) $x_1 = 2 - \frac{2\cdot 8 - 4\cdot 2 + 3}{6\cdot 4 - 4} = 1.45$

$$x_2 = 1.45 - \frac{2\cdot 1.45^3 - 4\cdot 1.45 + 3}{6\cdot 1.45^2 - 4} \approx 1.07$$

12. (A) Completing the square…

$$\int \frac{1}{-\sqrt{6v - v^2}}\,dv = \int \frac{1}{-\sqrt{9-(v-3)^2}} = -\arcsin\frac{x-3}{3} + C \equiv \arccos\frac{x-3}{3} + C$$

Because arcsin $\frac{x-3}{3}$ + arccos $\frac{x-3}{3}$ = $\frac{\pi}{2}$,

therefore -arcsin $\frac{x-3}{3}$ = arccos $\frac{x-3}{3}$ + C

13. (B) $f'(x) = 6x - 4, f'(3) = 14 \quad f(3) = 45$

$g'(x) = 4x, g'(2) = 1 \quad g(2) = -\frac{1}{4}$

Line tangent to f(x): $y = 14x + 3$

Line perpendicular to the tangent of g(x): $y = -x + \frac{7}{4}$

Intersect at $\left(\frac{-1}{12}, \frac{11}{6}\right)$

14. (A) $2\pi\int_0^2 x((-(x-2)^2 + 4) - (x^3 - x^2))dx = 2\pi\int_0^2 4x^2 - x^4 dx = 2\pi\frac{64}{15} = \frac{128\pi}{15}$

15. (D) $p = \frac{k}{V} \quad W = \int_1^5 \frac{800}{V} dV = 800\ln|V|\Big]_1^5 \approx 1287.55$

16. (D) $\frac{d}{dx}\left[\tan(x)\cot(x)\right] = \frac{d}{dx}[1] = 0$

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$$h'(x) = 3(g(x))^2 (f(x))^2 g'(x) + 2(g(x))^3 f(x) f'(x)$$

17. (B) $h'(4) = 3(g(4))^2 (f(4))^2 g'(4) + 2(g(4))^3 f(4) f'(4)$

$$= 3(5)^2 (8)^2 \frac{1}{4} + 2(5)^3 8 \cdot 2 = 11200$$

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18. (D) To be a probability function, the sum of all terms must equal 1:

$$\int_1^\infty p(x)dx = 1 = \int_1^\infty \frac{h^2}{x^4} dx = \frac{h^2}{3} \quad \Rightarrow \; h = \sqrt{3}$$

19. (C) Arc Length for a polar equation is: $\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

In 10 minutes, the ship travels

$\int_{\pi/4}^{\pi/3} \sqrt{(6\cos\theta)^2 + (-6\sin\theta)^2} d\theta = 6\theta\Big|_{\pi/4}^{\pi/3} = \frac{\pi}{2}$ which means it moves at a speed of $\frac{\pi}{20} \frac{units}{\min}$. The remaining distance is

$\int_{\pi/3}^{\pi/2} \sqrt{(6\cos\theta)^2 + (-6\sin\theta)^2} d\theta = 6\theta\Big|_{\pi/3}^{\pi/2} = \pi$, therefore it will take

$$\pi\, units \Big/ \frac{\pi}{20} \frac{units}{\min} = 20 \min$$

20. (B) $A(x) = \frac{2x^3 + 10x^2 - 48x}{x^3 - 8x^2 - 9x + 72} = \frac{2x(x+3)(x-8)}{(x-3)(x+3)(x-8)} = \frac{2x}{x-3}$

Therefore, it has 1 vertical and 1 horizontal asymptote (since x = -3 and x = 8 are removable discontinuities)

21. (A) If we make $u = x^{1/2}$ then $u^2 = x \Rightarrow 2u\,du = dx$

$\int \frac{1}{x^{1/2} + 2} dx = \int \frac{2u}{u+2} du = \int 2 - \frac{4}{u+2} du$ by polynomial division

$$\int 2 - \frac{4}{u+2} du = 2u - 4\ln|u+2| + C = 2\sqrt{x} - 4\ln\left|\sqrt{x} + 2\right| + C$$

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$x + y = 9 \quad \Rightarrow \quad y = 9 - x$

22. (C) $f(x,y) = xy^2 = x(9-x)^2 = f(x)$

$f'(x) = 3(x-9)(x-3) = 0 \qquad x = 3 \text{ or } 9$

Given $0 \le x \le 9$, we test x = 0,3 and 9

$f(0) = f(9) = 0$

$f(3) = 108$

Therefore, (3,6) produces a maximum of 108

23. (C) 10 ft/s = 60 ft/min

$x^2 + y^2 = z^2$ yields $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$ (total differentiation)

when x = 45 ft and z=51 ft, y=24 ft. Also, $\frac{dy}{dt} = 0$.

$$24ft \cdot 600\frac{ft}{\min} + 45ft \cdot 0\frac{ft}{\min} = 51ft \cdot \frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{4800}{17}\frac{ft}{\min} = 282.35\frac{ft}{\min}$$

24. (D) $f(x) = x^3e^{2x}$

$$f'(x) = (2x^3 + 3x^2)e^{2x} = 0$$

$$f''(x) = 2x(2x^2 + 6x + 3)e^{2x} = 0$$

$$x = 0, \frac{-3 \pm \sqrt{3}}{2}$$

Testing values between each interval, we find signs change at each x value, which means there are 3 inflection points.

25. (C) Since $\cos x, \sin x > 0 \; when\, 0 \le x \le \frac{\pi}{4}$, then we can ignore the absolute value.

$$\int_0^{\pi/4} \frac{\cos 2x}{|\cos x + \sin x|}\, dx = \int_0^{\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}\, dx = \int_0^{\pi/4} \cos x - \sin x \; dx = \sqrt{2} - 1$$

26. (C) Using the Shell Method:

$$2\pi\int_3^4 x(x^3 - 2x^2 - 1)dx = 2\pi\int_3^4 x^4 - 2x^3 - xdx$$

$$= 2\pi\left[\frac{x^5}{5} - \frac{x^4}{2} - \frac{x^2}{2}\right]_3^4 = 2573.99 \approx 2574$$

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27. (A) $x = 2\sin t$, $y = 3\cos t$ curvature = $\dfrac{|r' \times r''|}{|r'|^3}$

$r = \langle 2\sin t, 3\cos t \rangle$

$r' = \langle 2\cos t, -3\sin t \rangle$ $|r'| = \sqrt{4\cos^2 t + 9\sin^2 t}$

$r'' = \langle -2\sin t, -3\sin t \rangle$

$$|r' \times r''| = \begin{vmatrix} i & j & k \\ 2\cos t & 3\sin t & 0 \\ -2\sin t & -3\cos t & 0 \end{vmatrix} = \left| \langle 0, 0, -6\cos^2 t + 6\sin^2 t \rangle \right| = 6(\sin^2 t - \cos^2 t)$$

for $\left(1, \dfrac{3\sqrt{3}}{2}\right)$, t = $\dfrac{\pi}{6}$

$$\therefore \frac{|r' \times r''|}{|r'|^3} = \frac{\left|6(\sin^2 \frac{\pi}{6} - \cos^2 \frac{\pi}{6})\right|}{(4\cos^2 \frac{\pi}{6} + 9\sin^2 \frac{\pi}{6})^{1/2}} = \frac{2\sqrt{21}}{7} \approx 1.309$$

28. (D)

$$\frac{dz}{dt} = k(z - 15)$$

$$\int \frac{dz}{z - 15} = kdt$$

$$\ln(z - 15) = kt + C$$

$$\Rightarrow z = Ce^{kt} + 15$$

$for \ \ t = 0$

$18.3 = C + 15$

$\Rightarrow C = 3.3$

$for \ \ t = 2$

$19.3 = 3.3e^{2k} + 15$

$\Rightarrow k \approx 0.132346$

$for \ \ t = 6$

$z = 3.3e^{6 \cdot 0.132346} + 15$

$z \approx 22.3$

29. (B) $\displaystyle\int_0^6 x^2 e^{x/2} dx = \frac{b - a}{3(n - 1)}(f(x_1) + 4(x_2) + f(x_3))$

where $x_1 = 0, x_2 = 3, x_3 = 6$ since we increment by intervals of $\dfrac{b - a}{n - 1} = \dfrac{6 - 0}{3 - 1} = 3$.

$$\int_0^6 x^2 e^{x/2} dx = \frac{6-0}{3(3-1)}(0^4 e^0 + 4(3^2 e^{3/2}) + 6^2 e^3) \approx 884.420$$

30. (B) Half-life is calculated by the formula,

$n = n_0\left(\frac{1}{2}\right)^{t/h}$ where t is how long it has been, h is the half life, n is the current amount and n_0 is the original amount.

$$172 = 180\left(\frac{1}{2}\right)^{t/5730} \Rightarrow t = 5730\log_{.5}\left(\frac{172}{180}\right) \approx 375.821 \approx 376$$