Question 1
State Calculus Bowl
Mu Alpha Theta National Convention 2003

What is the sum of *A*, *B*, *C*, and *D* that makes the following function both continuous and differentiable for all *x*?

$$f(x)=\begin{cases} x^2+Ax+B & x<2 \\ \sqrt{x-1} & 2\le x\le 5 \\ x^2+Cx+D & x>5 \end{cases}$$

Question 2
State Calculus Bowl
Mu Alpha Theta National Convention 2003

A parabolic mirror is set up such that its base is on the origin, its reflective side is pointing towards the *y*-axis, and its shape is governed by the equation $y=x^2,\ -1\le x\le 1$. Light comes into this mirror from a source following the line $4x+3=0$. This light gets reflected to another side of the mirror along the line $Ax+By+C=0$ (where the greatest common factor of *A*, *B*, and *C* is 1) and then gets reflected away from the mirror on a line of the form $Dx+Ey+F=0$ (where the greatest common factor of *D*, *E*, and *F* is 1). What is the value of $ABC+DEF$?

Question 3
State Calculus Bowl
Mu Alpha Theta National Convention 2003

Let:

$$A=\int_0^1\left(1-x+x^2-x^3+\cdots\right)dx,$$

$$B=\int_0^1\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)dx, \text{ and}$$

$$C=\int_0^1\left(1-x^2+x^4-x^6+\cdots\right)dx.$$

What is the value of $e^A+2B+8C$?

Question 4
State Calculus Bowl
Mu Alpha Theta National Convention 2003

The function $y=A\cosh x+B\sinh x+C\cos x+D\sin x$ is a solution of the differential equation: $y^{(4)}-y=0$, (where *A*, *B*, *C*, and *D* are constants). For what values of *A*, *B*, *C*, and *D* (listed in that order) does *y* satisfy the initial conditions $y(0)=y'(0)=0$ and $y''(0)=y'''(0)=4$?

Question 5
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Mu Alpha Theta National Convention 2003

Evaluate the following:

$$\int_1^{e^{\pi/2}}\sin(\ln x)\,dx+\int_1^{e^{\pi/2}}\cos(\ln x)\,dx$$

Question 6
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I am interested in the region bounded by the graphs of $y=(x/\pi)^3$ and $y=(x/\pi)+\sin x$ for all *x*. Let *A* be the volume created by rotating this region about the *x*-axis, and let *B* be the volume created by rotating this region about the *y*-axis. What is the sum of $42A+15B$?

Question 7
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Mu Alpha Theta National Convention 2003

Evaluate the definite integral:

$$\int_{-\pi}^{\pi}\left[x^2\sin x-x^2\cos x-\tan^3 x\sec^4 x\right]dx$$

Question 8
State Calculus Bowl
Mu Alpha Theta National Convention 2003

Let *A* be the radius of curvature at the point (-3,0) on the graph of $16x^2+9y^2+96x+72y+144=0$, and let *B* be the curvature at the point $\left(-3+3\sqrt{2},0\right)$ on the graph of $16x^2-9y^2+96x-72y-144=0$. What is the value of $4|A|+\sqrt{41}|B|$?

Question 9
State Calculus Bowl
Mu Alpha Theta National Convention 2003

Suppose that at time t $(t>0)$ the position of a particle moving on the *x*-axis is $x=(t-1)(t-4)^4$. Let *A* be the smallest value of *t* for which the particle will be at rest, and let *B* be the fastest speed the particle goes while moving to the left (towards negative *x*). What the value of $125(A+B)$?

Question 10
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Mu Alpha Theta National Convention 2003

Use differentials to find an approximate value (leave answer as a fraction) for:

$$\ln(11/10)+\sqrt{24}+\sqrt[3]{124}+\sqrt[5]{33}$$

Question 11
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Let *A* be the volume of the solid in the first octant bounded by the coordinate planes, the plane $x=3$, and the parabolic cylinder $z=(1/4)-y^2$. Let *B* be the surface area of the same shape. What is $12(A+B)$?

Question 12
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Order the following functions from fastest growing to slowest growing as $x\to\infty$ (On your answer sheet, only mark the letters for th correct order).

A. $(x+1)^{x-1}$
B. $(x-1)^{x+1}$
C. 3^x
D. e^x
E. x^x

Question 13
State Calculus Bowl
Mu Alpha Theta National Convention 2003

A gardener wants to plant a flower bed in the shape of a circular sector of radius *r* and central angle *θ*. What is *r* and *θ*, if the area is fixed at 16 and the perimeter is a minimum? Give answer in the form $(r,\ \theta)$.

Question 14
State Calculus Bowl
Mu Alpha Theta National Convention 2003

The parametric equations for an astroid centered at the origin are:

$x=A\cos^3\theta$ and $y=A\sin^3\theta$

What is the sum of the area and the perimeter of this shape when it is drawn for $A=4$?

Question 15
State Calculus Bowl
Mu Alpha Theta National Convention 2003

Let $f(x)=\left[2x-(3/x)\right]^7$. What is the value of the constant term in the full expansion of $f'(x)$?

Mu Alpha Theta National Convention 2003
State Calculus Bowl Solutions

The correct answer to each question is given immediately after the question number in parentheses.

1. (33/2) To be continuous and differentiable, the $f(x)$ and $f'(x)$ must be the same for both sides of $x=2$ and $x=5$. This leads to $4+2A+B=1$, $4+A=1/2$, $2=25+5C+D$, and $1/4=10+C$. So $A=-3.5$, $B=4$, $C=-9.75$, and $D=25.75$. The sum is 16.5.

2. (-180) Light coming into a parabolic mirror parallel to the axis of symmetry will reflect toward the focus. The line $4x+3=0$ intersects the mirror at $(-3/4, 9/16)$. The focus of the mirror is at $(0, 1/4)$. So the line containing these two points is $5x+12y-3=0 \rightarrow ABC=-180$. Once this light gets to the other side of the mirror, it will reflect away from the mirror parallel to the axis of symmetry. Because of this, the *E* term will be 0, and $DEF=0$. So $ABC+DEF=-180$.

3. $(2e+2\pi)$ $A=\int_0^1(1-x+x^2-x^3+\cdots)dx=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=(\ln(x+1))]_0^1=\ln 2\rightarrow e^A=2$

$B=\int_0^1\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)dx=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=(e^x-1]_0^1=(e-1)\rightarrow 2B=2e-2$

$C=\int_0^1(1-x^2+x^4-x^6+\cdots)dx=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=(\tan^{-1}x]_0^1=\frac{\pi}{4}\rightarrow 8C=2\pi$. So $e^A+2B+8C=2e+2\pi$.

4. (2,2,-2,-2) $y=A\cosh x+B\sinh x+C\cos x+D\sin x=0=A+C$; $y'=A\sinh x+B\cosh x-C\sin x+D\cos x=0=B+D$; $y''=A\cosh x+B\sinh x-C\cos x-D\sin x=4=A-C$; and $y'''=A\sinh x+B\cosh x+C\sin x-D\cos x=4=B-D.\rightarrow(2,2,-2,-2)$

5. $(e^{\pi/2})$ Let $u=\ln x\rightarrow e^u=x\rightarrow e^u du=dx$. This changes the integrals to $\int e^u\sin u+e^u\cos u\,du$. This can be solved by parts and then replacing *x* gives: $(x\sin(\ln x)]_1^{e^{\pi/2}}=e^{\pi/2}$.

6. $(8\pi^3+118\pi^2+168\pi)$ The regions intersect at $x=-\pi$, 0, and π. The region is an odd function, so we can calculate only the part fo $x>0$ and then double it. $A=2\int_0^\pi\pi\left[\left(\frac{x}{\pi}+\sin x\right)^2-\left(\frac{x^3}{\pi^3}\right)^2\right]dx=2\left(\frac{x^3}{3\pi}-2x\cos x+2\sin x-\frac{x^7}{7\pi^5}\right]_0^\pi+\pi^2=\frac{29\pi^2+84\pi}{21}\rightarrow 42A=58\pi^2+168\pi$.

$B=2\int_0^\pi 2\pi x\left(\frac{x}{\pi}+\sin x-\frac{x^3}{\pi^3}\right)dx=\left(\frac{4}{3}x^3-4\pi x\cos x+4\pi\sin x-\frac{4x^5}{5\pi^2}\right]_0^\pi=\frac{8\pi^3+60\pi^2}{15}\rightarrow 15B=8\pi^3+60\pi^2$. So $42A+15B=8\pi^3+118\pi^2+168\pi$.

7. (4π) The trick to this question is to notice that the first and third terms in the integrand are odd functions. Thus they contribute nothing to the definite integral. Thus the question reduces to $-\int_{-\pi}^{\pi}x^2\cos x\,dx$. This integrates to $(-x^2\sin x-2x\cos x+2\sin x]_{-\pi}^{\pi}=4\pi$.

8. (381/41) Curvature is defined as $\kappa=|y''|/\left(1+(y')^2\right)^{3/2}$. Radius of curvature is just the inverse. For the first problem, I found it simpler to convert the equation to the form: $16(x+3)^2+9(y+4)^2=144\rightarrow 32(x+3)dx+18(y+4)dy=0$. Then $dy/dx=-16(x+3)/[9(y+4)]=-4(x+3)/\left(3\sqrt{-x^2-6x}\right)$ when you substitute for *y*. The second derivative is then $d^2y/dx^2=12/(-x^2-6x)^{3/2}$. Plugging in for the radius of curvature, $1/\kappa=(7x^2+42x+144)^{3/2}/324$. For $x=-3$, this gives $1/\kappa=9/4\rightarrow 4|A|=9$. For the second problem you get, $16(x+3)^2-9(y+4)^2=144\rightarrow 32(x+3)dx-18(y+4)dy=0$. Then $dy/dx=16(x+3)/[9(y+4)]=4(x+3)/\left(3\sqrt{x^2+6x}\right)$ when you substitute for *y*. The second derivative is then $d^2y/dx^2=-12/(x^2+6x)^{3/2}$. Plugging in for the curvature, $\kappa=324/(25x^2+150x+144)^{3/2}$. For $x=-3+3\sqrt{2}$, this gives $\sqrt{41}|\kappa|=12/41=\sqrt{41}|B|$. So $4|A|+\sqrt{41}|B|=9+12/41=381/41$.

9. (2387) $x=(t-1)(t-4)^4\rightarrow v=4(t-1)(t-4)^3+(t-4)^4\rightarrow v=(t-4)^3(5t-8)\rightarrow a=3(5t-8)(t-4)^2+5(t-4)^3$ $\rightarrow a=(t-4)^2(15t-24+5t-20)=(t-4)^2(20t-44)$. The first time the particle will be at rest is at $t=8/5=A$. The particle moves to the left for *t* between 8/5 and 4. It reaches zero acceleration during this period at *t* of 11/5. It's speed at this point is $2187/125=B$. So $125(A+B)=125(8/5+2187/125)=2387$.

Mu Alpha Theta National Convention 2003
State Calculus Bowl Solutions, Page 2

10. (14399/1200) $y=\ln x \to dy = dx/x$. For $x=1$ and $dx=1/10$, $y=0$ and $dy=1/10$. $y=\sqrt{x} \to dy = dx/\left(2\sqrt{x}\right)$. For $x=25$ and $dx=-1$, $y=5$ and $dy=-1/10$. $y=x^{1/3} \to dy = dx/\left(3x^{2/3}\right)$. For $x=125$ and $dx=-1$, $y=5$ and $dy=-1/75$. $y=x^{1/5} \to dy=dx/\left(5x^{4/5}\right)$. For $x=32$ and $dx=1$, $y=2$ and $dy=1/80$. Adding the y's and the dy's gives $14399/1200$.

11. $\left[32+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right) \text{ OR } 32+9\sqrt{2}+9\sinh^{-1}1\right]$ $A=\int_0^3 dx\int_0^{1/2} dy \int_0^{1/4-y^2} dz = 3\left(y/4-y^3/3\right]_0^{1/2} = 1/4 \to 12A=3$ For surface area, let's get all the easy sides first. The side on $y=0$ has area 3/4. The side on $z=0$ has area 3/2. The areas on $x=0$ and $x=3$ are the same: $\int_0^{1/2} 1/4-y^2 dy = 1/12$...times two is 1/6. Now for the surface area on the curved side. You need to calculate the length of the parabolic arc, L, and then multiply by 3 (the width) to find the area. $L=\int_0^{1/2}\sqrt{1+z'^2}\,dz = \int_0^{1/2}\sqrt{1+4y^2}\,dy$. Let $y=\frac{1}{2}\tan\theta \to dy = \frac{1}{2}\sec^2\theta d\theta$. This gets $L=\int_0^{\pi/4}\left(\sec^3\theta\right)/2\,d\theta$, which can be integrated by parts for $u=.5\sec\theta$ and $dv=\sec^2\theta d\theta$ to get $L=.25\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4}$ $=0.25\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$. Thus the area is three times this, and $12B$ is $29+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right)$. So the answer is: $12(A+B)=32+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right)$ OR $12(A+B)=32+9\sqrt{2}+9\sinh^{-1}1$ (same answer, just derived differently).

12. (B,E,A,C,D) Compare two functions by dividing them and looking at the limit as x goes to infinity. First off, C and D are much slower than A, B, or E. Now let's compare them. $\lim_{x\to\infty}(3/e)^x=\infty$, so C is faster than D. Now let's compare A and E.
$\lim_{x\to\infty}\frac{(x+1)^{x-1}}{x^x}=\lim_{x\to\infty}\frac{\left[(x+1)/x\right]^x}{x+1}=\frac{e}{\infty}=0$, so E is faster than A. Now let's compare B and E: $\lim_{x\to\infty}\frac{(x-1)^{x+1}}{x^x}=\lim_{x\to\infty}(x-1)\left(\frac{x-1}{x}\right)^x=\infty e=\infty$, so B is faster than E. Thus the answer is B, E, A, C, and D.

13. [(4,2)] $Area=r^2\theta/2$ and the $Perimeter=r\theta+2r=2Area/r+2r=32/r+2r \to dP=-32/r^2+2=0\to r=4$. Plugging back into the area equation gives $\theta=2$. So the required answer is (4,2).

14. (6π+24) The area of an astroid as given in the problem is $3\pi A^2/8$. The perimeter is $6A$. Plugging in the value of A given, you have that the Area + Perimeter $=6\pi+24$. The derivation involves lots of using the sum of cosine squared and sine squared, as well as integration by parts, and the symmetry of the system (all of the calculations can be done in the first quadrant alone and then multiplied by 4).

15. (-15120) The constant term in the derivative will be the term that had a degree of 1 in $f(x)$. This is the fourth term. The rth term in the expansion of $(a+b)^n$ is $\binom{n}{r-1}(a)^{n-r+1}(b)^{r-1}=\binom{7}{3}(2x)^4\left(-\frac{3}{x}\right)^3=-15120x$. So the value of the constant term in the full expansion of $f'(x)$ is -15120.