Mu Alpha Theta National Convention 2003
Matrices and Vectors Alpha Solutions

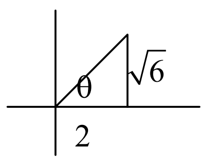
1. B $2x+4=4, x=0; 2-4y=7, y=-\frac{5}{4}, x+y=-\frac{5}{4}$

2. A $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 0 \\ 3 & 4 & -2 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}; \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 4.0 \\ 1.5 \\ 7.5 \end{pmatrix} \rightarrow r-5=2.5$

3. C Rank=number of pivots $\begin{pmatrix} 2 & 4 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

pivots = 3

4. D



$\tan\theta = \frac{\sqrt{6}}{2}, \theta = arc\frac{\sqrt{6}}{2} \approx 50.8$

5. E We want dot product of third row of 1st matrix and fourth column of 2nd matrix.

$\begin{bmatrix} 6 & 28 & 496 & 8128 \end{bmatrix} \circ \begin{bmatrix} 7 & 16 & 13 & 5 \end{bmatrix} = 47578$

6. A $\left\|\vec{a}+\vec{b}\right\|^2 = \left(\vec{a}+\vec{b}\right)\circ\left(\vec{c}+\vec{b}\right) = \left\|\vec{a}\right\|^2 + 2\vec{a}\vec{b} + \left\|\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 = \vec{a}\circ\vec{b} = 0$. This means they're orthogonal.

7. B By definition, a singular matrix has a determinant of 0.

8. D $\cos\theta = \frac{\vec{u}\circ\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|} = \frac{24+60}{(5)(17)} = \frac{84}{85}, \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-(84/85)^2}} = \frac{85}{13}$

9. D $\begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-2 & 3 \\ 0 & 0 & x-5 \end{vmatrix} = 0$, lower triangular, so determinant is just product of main diagonal entries, $(x-1)(x-2)(x-5)=0, x=1,2,5,$ sum is 8.

10. C $\frac{(x\circ y)(y\circ z)(x\circ z)}{(y\circ z)^2\left\|x\right\|^2} = \frac{(x\circ y)(x\circ z)}{(y\circ z)(x\circ x)} = \frac{400}{427} = \frac{(4+10+18)(7+16+27)}{(28+40+54)(1+4+9)}$

11. A $\left|ABC\right| = \left|A\right|\left|B\right|\left|C\right|, 12\bullet 51 = (32-12)(17)\left|C\right| \rightarrow \left|C\right| = 1.8 = \frac{9}{5} \rightarrow m+n=14$

12. B $\left.\begin{array}{l} \left\|\vec{x}+\vec{y}\right\|^2 = \left\|\vec{x}\right\|^2 + 2\vec{x}\vec{y} + \left\|\vec{y}\right\|^2 \\ \left\|\vec{x}-\vec{y}\right\|^2 = \left\|\vec{x}\right\|^2 - 2\vec{x}\vec{y} + \left\|\vec{y}\right\|^2 \end{array}\right\}$ $\left\|\vec{x}+\vec{y}\right\|^2 - \left\|\vec{x}-\vec{y}\right\|^2 = 4x\circ y$

$64 - \left\|\vec{x}-\vec{y}\right\|^2 = 28, \left\|\vec{x}-\vec{y}\right\|^2 = 36, answer = 6$

13. B The matrix represents a counter-clockwise rotation of $\frac{\pi}{6}$ about the origin repeated 2002 times. Thus, it's the same as a single rotation of $\frac{\pi}{6}(2002)=\frac{1001\pi}{3}$. This is coterminal to $\frac{5\pi}{3}$ so the answer is $\begin{pmatrix}\cos\frac{5\pi}{3} & -\sin\frac{5\pi}{3}\\ \sin\frac{5\pi}{3} & \cos\frac{5\pi}{3}\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$.

14. D Clearly a=7. Dot product equals 0 if orthogonal: $\begin{bmatrix}3 & b & 7\end{bmatrix}\circ\begin{bmatrix}17b & -2 & 7\end{bmatrix}=0$
$51b-2b+49=0\rightarrow b=-1, a-b=7-(-1)=8$

15. D $\begin{pmatrix}4 & 6\\ 6 & 0\end{pmatrix}^T=\begin{pmatrix}4 & 6\\ 6 & 0\end{pmatrix}\leftarrow symmetric, \begin{pmatrix}7 & 1\\ 4 & 4\end{pmatrix}^T=\begin{pmatrix}7 & 4\\ 1 & 4\end{pmatrix}\neq\begin{pmatrix}7 & 1\\ 4 & 4\end{pmatrix}\leftarrow not$

$\begin{pmatrix}2 & 5\\ 5 & 8\end{pmatrix}^T=\begin{pmatrix}2 & 5\\ 5 & 8\end{pmatrix}\leftarrow symmetric, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^T=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=-\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\leftarrow skew\ symmetric$

$\begin{pmatrix}3 & 1\\ -1 & 9\end{pmatrix}^T=\begin{pmatrix}3 & -1\\ 1 & 9\end{pmatrix}\leftarrow not, m=2, n=1, (m-n)^m=(2-1)^2=1$

16. C $\begin{pmatrix}8 & -21\\ 3 & -8\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}3\\ 1\end{pmatrix}=1\bullet\begin{pmatrix}3\\ 1\end{pmatrix}\leftarrow so\begin{bmatrix}3 & 1\end{bmatrix}$ is an eigenvector. Easy to check the other one's don't work.

17. D Pick $(0,-7,5)$ as origin.

$\vec{x}=\begin{bmatrix}1 & 13 & -17\end{bmatrix}, \vec{y}=\begin{bmatrix}2 & 7 & -1\end{bmatrix}, \vec{x}\otimes\vec{y}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 13 & -17\\ 2 & 7 & -1\end{vmatrix}=\vec{i}(106)-\vec{j}(33)+\vec{k}(-19)$

$\left\|\vec{x}\otimes\vec{y}\right\|=\sqrt{106^2+33^2+19^2}=\sqrt{12686}, ans=\frac{1}{2}\left\|\vec{x}\otimes\vec{y}\right\|=\frac{\sqrt{12686}}{2}$

18. B $proj_{\vec{v}}\vec{u}=\frac{\vec{u}\vec{v}}{\left\|\vec{v}\right\|^2}\vec{v}=\frac{(-14+0-30)}{(49+0+25)}\begin{bmatrix}-7 & 0 & -5\end{bmatrix}=\frac{-22}{37}\begin{bmatrix}-7 & 0 & -5\end{bmatrix}$

19. A A is diagonal, so $A^i=\begin{pmatrix}1^i & 0\\ 0 & 2^i\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 2^i\end{pmatrix}. \sum_{i=0}^{n}A^c=\begin{pmatrix}1+1+...+1 & 0\\ 0 & 1+2+4+...2^n\end{pmatrix}$

$$=\begin{pmatrix} n+1 & 0 \\ 0 & 2^{n+1}-1 \end{pmatrix}$$. Trace = sum of main diagonal entries = $2^{n+1}+n$

20. D Normal vectors perpendicular means planes are perpendicular.
Since $\begin{bmatrix}5 & -7 & 3\end{bmatrix} \circ \begin{bmatrix}5 & 1 & -6\end{bmatrix} = 0$, the answer is D.

21. C For AB to exist, $x^3+7x=10x^2-18 \rightarrow (x+2)(x-2)(x-9)=0, x=-1,2,9.$
Be careful! -1 and 2 will give negative dimensions! Only x=9 works.

22. A $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta = (5)(3)\sin\theta \le 15\sin\frac{\pi}{2} = 15$. Max length when $\theta = \frac{\pi}{2} \rightarrow 15.$

23. C I. $\begin{vmatrix} 8 & 9 & 1 \\ 4 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} \xrightarrow[\text{2. } \frac{R_1}{3} \text{ 3. } \frac{R_2}{6}]{\text{1. transpose}} \begin{vmatrix} \frac{8}{3} & \frac{4}{3} & 1 \\ \frac{3}{2} & \frac{1}{3} & 1 \\ 1 & 1 & 1 \end{vmatrix} \rightarrow$ Area will be $\frac{1}{3}\bullet\frac{1}{6}=\frac{1}{18}th$ of the original.

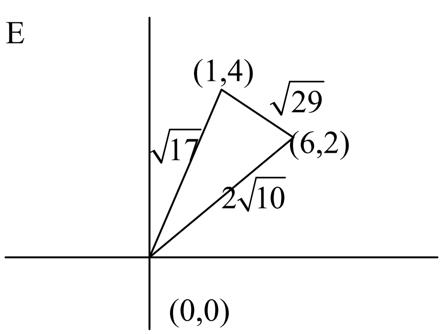
II. $\begin{vmatrix} 7 & 6 & 1 \\ 5 & 13 & 1 \\ 1 & 1 & 1 \end{vmatrix} \rightarrow transpose \begin{vmatrix} 7 & 5 & 1 \\ 6 & 13 & 1 \\ 1 & 1 & 1 \end{vmatrix} \rightarrow$ doesn't change abs value of determinant.

III. $\begin{vmatrix} 13 & 2 & 1 \\ -5 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 13 & 1 \\ 4 & -5 & 1 \\ 0 & 0 & 1 \end{vmatrix} \rightarrow$ doesn't change abs value of determinant

IV. $\begin{vmatrix} 2 & 4 & 1 \\ -8 & 0 & 1 \\ 1 & 5 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & -8 & 1 \\ -4 & 32 & 1 \\ 1 & 1 & 1 \end{vmatrix} \rightarrow$ doesn't change abs value of determinant

24. C $\left|\begin{pmatrix} c & 7 & -6 \\ 3 & 1 & -2 \\ 1 & c & 2 \end{pmatrix}\right| = 0, (c-5)^2 = 0, c=5$, for linear dependence

25. E



$$Area = \pm\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 4 & 1 \\ 6 & 2 & 1 \end{vmatrix} = \pm\frac{1}{2}(2-24) = 11$$

26. A $\left\|\vec{u}\right\| = \sqrt{\begin{matrix}16e^{8t}\cos^2 3t - 24e^{8t}\cos 3t\sin 3t + 9e^{8t}\sin^2 3t \\ + \\ 16e^{8t}\sin^2 3t + 24e^{8t}\cos 3t\sin 3t + 9e^{8t}\cos^2 3t\end{matrix}} = \sqrt{16e^{8t} + 9e^{8t}} = 5e^{4t}$

strictly increasing so next value at t=5 and min at t=0, Ans = $\dfrac{f(5)}{f(0)} = \dfrac{5e^{20}}{5} = e^{20}$

27. B We need $d \mid (n^2+1)$ and $d \mid (n^2+2n+2) \rightarrow d \mid (n+1)^2+1$. Then $d \mid ((n+1)^2+1-(n^2+1)) \rightarrow \mid (2n+1)$. Thus, $d \mid (2n+1)^2 \rightarrow d \mid (4n^2+4n+1) \rightarrow d \mid (4(n^2+2n+2)-(4n^2+4n+1))$ which means $d \mid (4n+7).)$ So $d \mid \left((4n+7)-2(2n+1)\right)$ or d|5.

28. E $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 20 \\ 7 & 27 & 37 \\ -6 & -10 & 20 \end{pmatrix} \rightarrow \begin{pmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{pmatrix} = \begin{pmatrix} -1 & 2 & 7 \\ 0 & -5 & 4 \\ 3 & 6 & 9 \end{pmatrix}$,

$\vec{x}\circ\vec{x} + \vec{y}\circ\vec{y} + \vec{z}\circ\vec{z} = 1+4+49+0+25+16+9+36+81 = 221$

29. B $\dfrac{a_1}{b_1} = \dfrac{1}{1}, \dfrac{a_2}{b_2} = \dfrac{5}{2}, \dfrac{a_3}{b_3} = \dfrac{13}{6}, \dfrac{a_4}{b_4} = \dfrac{29}{14}, \ldots$

n	$\lvert x \rvert$
1	1
2	$1.5 = 1+\frac{1}{2}$
3	$1.75 = 1+\frac{1}{2}+\frac{1}{4}$
4	$1.875 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

This can be proved but kind of time-consuming. As $n \rightarrow \infty, \lvert x \rvert \rightarrow 1+\frac{1}{2}+\frac{1}{4}+\ldots = 2$

30. C A basis for $\mathbb{R}^4$ needs 4 distinct vectors, so A, B, and D are out of the question.

In fact, C is a basis because $\begin{vmatrix} 7 & 8 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & -1 & -2 & -1 \end{vmatrix} = -10 \neq 0$ so it's linearly independent and its span is $\left[7a+8b+c+3d, a, 2a+b+c, -b+2c-d\right]$, which covers all 4-vectors.