Differential Equations Topic Test: Complete Solutions

Note: For each problem, where there is no choice (e), assume (e) none of the above.

1. State the order of the differential equation: $(y')^3 = \sin x$

   a) 1st b) 2nd c) 3rd d) 4th

answer: a

solution: Ignore the 3rd power. The equation remains of order 1.

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2. The solution to the differential equation $x\,dy - y\,dx = 0$ is

   a) $y = e^x + C$ b) $y = Cx$ c) $y = x$ d) cannot be solved

answer: b

solution:

$x\,dy - y\,dx = 0$

$\int \frac{dy}{y} = \int \frac{dx}{x} \rightarrow \ln|y| = \ln|x| + C \rightarrow y = e^{\ln x + C} \rightarrow y = e^{\ln x} \cdot e^C \rightarrow y = Cx$

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3. Solve $x\cos x\,dx + (1 - 6y^5)\,dy = 0$; The graph passes through $(\pi, 0)$

   a) $y^6 - y = \cos x - x\sin x + C$ b) $x\sin x - \cos x + \pi = y^6 - y$

   c) $y^6 - y = x\sin x + \cos x + 1$ d) no real valued solution exists

answer: c

solution:

$\int x\cos x\,dx = \int (6y^5 - 1)\,dy \rightarrow$ (integrate by parts on left) $x\sin x + \cos x + C = y^6 - y \rightarrow$

$\pi(\sin \pi) + \cos \pi + C = 0 \rightarrow 0 + -1 + C = 0 \rightarrow C = 1$

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4. Solve the differential equation: $(x^2 - xy + y^2)\,dx - xy\,dy = 0$

   a) $xy = Ce^{\frac{1}{x-y}}$ b) $(y - x)e^{\frac{y}{x}} = C$ c) $x = Ce^{(2x-y)}$ d) $C = \frac{2x}{2y-1}$

answer: b

solution:

Use y = vx $\rightarrow$ v + x $\frac{dv}{dx} = \frac{x^2 - x^2v + v^2x^2}{x(vx)} \rightarrow \frac{dv}{dx} = \frac{1}{x}\left(\frac{1-v}{v}\right) \rightarrow \frac{v\,dv}{1-v} = \frac{dx}{x} \rightarrow$

$\ln|x| + v + \ln|v-1| = \ln|C| \rightarrow x(v-1)e^v = C \rightarrow x\left(\frac{y}{x} - \frac{1}{1}\right)e^{\frac{y}{x}} = C \rightarrow (y - x)e^{\frac{y}{x}} = C$

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5. Solve the DE : $(x + y)y' + (y + 3x) = 0$

   a) $xy + \frac{3}{2}x^2 + \frac{1}{2}y^2 = C$ b) $3x^2 + y^2 = C$ c) $1 + \frac{3}{2}x^2 = \frac{y^2}{2}$ d) $C = x^3 - y^2 - xy$

answer: a

solution:

$(y + 3x)\,dx + (x + y)\,dy = 0 \rightarrow \frac{2F}{2x} = y + 3x \rightarrow \int (y + 3x)\,dx = xy + \frac{3}{2}x^2 + C_1 \rightarrow$

$\int (x+y)dy$ = xy + $\frac{y^2}{2}$ + C_2 $\rightarrow$ $\int (y+3x)dx$ = xy + $\frac{3}{2}x^2$ + g(y) $\rightarrow$

$\frac{\partial}{\partial x}\left(xy+\frac{3}{2}x^2+g(y)\right)=x+y$ $\rightarrow$ $g'(y)=y \Rightarrow g(y)=\frac{y^2}{2}+C$ $\rightarrow$ so xy + $\frac{3}{2}x^2$ + $\frac{y^2}{2}$ = C

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6. Solve the DE $(1+3x\sin y)dx - x^2\cos y\,dy = 0$

   a) $\frac{4}{x}\cos y = Cx^3 - 1$ b) $3x\sin y = Cx^2 - \ln|x| - 2$ c) $4x\sin y = Cx^4 - 1$ d) not solvable

answer: c
solution: Let u = siny

(1 + 3xu) dx - $x^2\,dy$ = 0

dx + 3x$u\ dx$ - $x^2\ du$ = 0

$\frac{dx}{x^2dx} = \frac{x^2du-3xudx}{x^2dx}$

$\frac{1}{x^2} = \frac{du}{dx} - \frac{3u}{x}$

$e^{\ln|x^{-3}|} = \frac{1}{x^{-3}}$

so integ. fac = $e^{\int \frac{-3}{x}dx}$
mult by x^{-3}

$x^{-3}\frac{du}{dx} - \frac{3}{x^4}u = \frac{1}{x^5}$

$\frac{d}{dx}\left(x^{-3}u\right) = x^{-3}\frac{du}{dx} - \frac{3}{x^4}u$ so

$\int \frac{d}{dx}\left(x^{-3}u\right) = \int \frac{1}{x^5}$

$x^{-3}u = -\frac{1}{4}x^{-4} + \frac{1}{4}C$

$4xu = Cx - 1$ replace u

$4x\sin y = Cx^4 - 1$

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7. What is the velocity of a projectile at an altitude of 8000 feet after it was fired directly upward from the ground with a muzzle velocity of 1000 feet per second? (g = 32ft/sec)

   a) 698.570 b) 770.366 c) 1229.634 d) 1698.570

answer: a
solution: see end

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8. A certain type of glass is such that a slab 1 inch thick absorbs one-quarter of the light which starts to pass through it. How thin must a pane be made to absorb only 1% of the light? (all answers are in inches).
   a) 0.007 b) 0.015 c) 0.028 d) 0.035

answer: d
solution: see end

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9. A certain radioactive material loses mass at a rate proportional to the mass present. If the material has a half-life of 30 minutes, what percent of the original mass is expected to remain after 0.9 hours?
   a) 8% b) 29% c) 47% d) 98%

Answer: b

Solution:

$$M = M_0 e^{rt} \rightarrow \frac{1}{2} = e^{rt} \rightarrow \frac{1}{2} = e^{r\left(\frac{1}{2}\right)} \rightarrow$$

$$\ln(0.5) = 0.5r \rightarrow -1.386 = r \quad so \rightarrow$$

$$\frac{M}{M_0} = e^{-1.386(0.9)} \approx 0.287 \ or \ 28.7\%$$

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10. If the marginal cost (y) of producing a certain item (x) is $\frac{dy}{dx} = 3 + x + \frac{e^{-x}}{4}$, what is the cost of producing one item if there is a fixed cost of $4.00?
    a) $6.76 b) $7.66 c) $8.16 d) $9.26

answer: b

solution:

$$y = \int \left(3 + x + \frac{e^{-x}}{4}\right) dx \rightarrow = 3x + \frac{x^2}{2} - \frac{e^{-x}}{4} + C$$

$$y = 4 \ \& \ x = 0$$

$$y = 4 = 0 + 0 - \frac{1}{4} + C = \frac{17}{4}$$

$$so \ \ x = 1 \Rightarrow 3 + \frac{1}{2} - \frac{1}{4e} + \frac{17}{4} \approx \$7.66$$

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11. Solve $y'' - y' - 2y = 0$.
    a) $y = C_1 e^{2x} + C_2 e^{-x}$ b) $y = C_1 e^{x} + C_2 e^{-2x}$ c) $y = C_1 e^{-x} + C_2 e^{-2x}$ d) not solvable

answer: a

solution: see end

12. What is the time required for one dollar to double when invested at the rate of 5% per annum compounded continuously?
    a) 0.139 yrs b) 1.386 yrs c) 13.863 yrs d) 138.629 yrs

answer: c

solution: $A = A_0 e^{rt} \rightarrow \text{since } A = 2A_0 \rightarrow 2 = 1e^{.05t} \rightarrow t \approx 13.863 \text{ yrs}$

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13. Solve the differential equation: $\frac{dy}{dx} = 3x^2$.

    $a)$ $y = x^3 + c$ b) y= 6x +c c) $y = 3x^3 + c$ d) $y = 3x^2 y + c$
    $E)$ NOTA

answer: a

solution: see end

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14. The temperature inside a house is 70° F. A thermometer is taken from the house and placed outside. The outside air is 10° F. After 3 minutes, the thermometer reads 25° F. What is the thermometer temperature after 7 minutes?
   a) 19° F b) 12° F c) 9° F d) 7° F
answer: b
solution: see end

15. A pipe 10 cm in diameter contains steam at 100° C. It is covered with asbestos 5 cm thick. The thermal conductivity, k, is 0.00060 cal/cm deg sec. The outside surface is at 30° C. Find the heat loss per hour from a meter length of pipe. (answers are in cal/hr)
   a) 380 b) 38,500 c) 138,000 d) 1,380,000
answer: d
solution: at end

16. What integrating factor would make the differential equation $2\left(y-4x^2\right)dx + xdy = 0$ exact?
   a) xy^2 b) $\frac{y}{x}$ c) x^2y d) x^2
answer: d
solution:
$2(y - 4x^2)\,dx + x\,dy = 0$
÷ by x
$\left(\frac{2}{x}y - 8x\right) dx + dy = 0$
$\frac{\partial M}{\partial y} = \frac{2}{x} \qquad \frac{\partial n}{\partial x} = 0$
so int. factor $\Rightarrow e^{\int \frac{2}{x}dx} = e^{\ln\left|x^2\right|} = x^2$

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17. Water flows down a river at the rate $9 + t^{\frac{3}{2}}$ million ft^3/day, t days after a rain. How much water will flow past a given point during the first 4 days after a rain? (answer in million ft^3)
   a) 39.4 b) 48.8 c) 61 d) 116
answer: b
solution:
$$\frac{dw}{dt} = 9 + t^{\frac{3}{2}}$$
$$W = \int_0^4 \left(9 + t^{\frac{3}{2}}\right)dt$$
$$W = \left(9t + \frac{2}{5}t\frac{5}{2}\right)\Bigg|_0^4$$
$$36 + \frac{64}{5} \simeq 48.8$$

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18. A solution of the differential equation 2ydy = xdx is

a) $x^2 - 2y^2 = 8$
b) $x^2 + 2y^2=8$
c) $2y^2 = -x^2$
d) $x^2 - 8y^2=0$
e) $x^2=16-2y^2$

answer: a

solution:

$$2ydy = xdx$$
$$2\left(y^2 = \frac{1}{2}x^2 + C\right)$$
$$2y^2 = x^2 + C$$
$$-x^2 + 2y^2 = C$$
$$x^2 - 2y^2 = C \text{ or } x^2 - 2y^2 = 4$$

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19. If a car accelerates from 0 to 70 mph in 10 sec, what distance does it travel in those 10 sec? (assume acceleration is constant and 60 mph=88ft/sec)

a) 51 ft b) 513 ft c) 616 ft d) 1027 ft

answer: b

solution:

$$\frac{70-0}{10} = \frac{70mph}{10\sec} \times \frac{88ft/s}{60mph} = 10.2\overline{6}$$
$$\int 10.2\overline{6}dt = 10.2\overline{6}t + C_1 = v(t)$$
$$t = 0, v = 0 \Rightarrow C_1 = 0$$
$$\int 10.2\overline{6}tdt = s(t) = 5.133t^2 + C_2$$
$$t = 0, s = 0 \Rightarrow C_2 = 0$$
$$s(t) = 5.133t^2$$
$$t = 10 \quad s = 513ft$$

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20. The growth size of an animal population at time t is denoted by $\frac{dp}{dt}$ = 0.002P (1000 – P). The population is growing fastest

a) initially
b) at the carrying capacity
c) when P=500
d) when $\frac{d^2P}{dt^2} > 0$

answer: c

solution: $\frac{dP}{dt}$ grows fastest when its derivative = 0 and when $\frac{dP}{dt}$ is concave down

$$\frac{dP}{dt} = 2P - 0.002P^2$$

$$\frac{d^2P}{dt^2} = 2 - 0.004P = 0 \quad when\ P = 500$$

$$\frac{d^3P}{dt^3} = -0.004 \Rightarrow \frac{dP}{dt}\ is\ concave\ down\ for\ all\ P$$

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21. Functions *g* and *h* are twice differentiable such that h(x)= ln(g(x)) and $h^{//}(x) = f(x)/(g(x))^2$
    Find f(x).
    a) $g(x)g^{//}(x)- 2g^{/}(x)$
    b) $g(x)g^{//}(x) - g^{/}(x)$
    c) $g(x)[g^{//}(x)]^2 - g^{/}(x)$
    d) $g(x)g^{//}(x) - [g^{/}(x)]^2$

answer: d
solution: see end

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22. Given $\frac{ds}{dt} = t^2 - t - 1$. If s = 0 when t=1, then what is the value of s when $t = 0$?

    a) $\frac{7}{6}$ b) $\frac{8}{7}$ c) $-\frac{4}{5}$ d) $\frac{1}{2}$

answer: a

solution: $\int ds = \int \left(t^2 - t - 1\right) dt \;\rightarrow\; s = \frac{t^3}{3} - \frac{t^2}{2} - t + C$

$s = 0,\ t = 0\ \ so\ 0 = \frac{1}{3} - \frac{1}{2} - 1 + C \rightarrow C = \frac{7}{6}$

$for\ t = 0\ \ s = \frac{7}{6}$

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23. Find the general solution for $y' + 2y = x^2$
    a) cannot be done
    b) $y = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$
    c) $y = 2x^2 - 8x + 19 + Ce^{-2x}$
    d) $y = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$

answer: b
solution: see end

24. The motion of a particle on the x-axis has acceleration $\frac{d^2x}{dt^2} = t^2 - 2t$. It is stationary at 1 when t = 1. Find $12x(t)$.

a) $t^4 + 4t^3$
b) $t^4 - 4t^3 + 8t + 7$
c) $4t^4 + 8t^3$
d) $t^4 - 4t^3 + 15t^2$

answer: b
solution: see end

25. The general solution of $x\,dy = y\,dx$ is a family of a) circles b) parabolas c) hyperbolas d) lines passing through the origin

answer: d

solution: $x\;dy = y\;dx$
$\int \frac{dy}{y} = \int \frac{dx}{x}$ → $\ln|y| = \ln|x| + C \rightarrow y = Cx$

26. If radium decomposes at a rate proportional to the amount present, then the amount R left after t years, if R_0 is present initially and k is a negative constant of proportionality, is given by

a) $R = R_0 kt$ b) $R = R_0\, e^{kt}$ c) $R = R_0 + \frac{1}{2}kt^2$ d) $R = e^{R_0 kt}$

answer: b

solution: $\frac{dR}{dt} = cR \rightarrow \ln R = ct + K \rightarrow R = Ke^{ct}$

since K= R_0 when t =0, then $R = R_0\, e^{ct}$

27. Given $\frac{ds}{dt} = \sin^2\left(\frac{\pi}{2}s\right)$ when t = 0, and s = 1. Find t when $s = \frac{3}{2}$.

a) $\frac{1}{2}$ b) $\frac{\pi}{2}$ c) 1 d) $\frac{2}{\pi}$

answer: d

solution: $\int \csc^2\left(\frac{\pi}{2}s\right)ds = \int dt$

$$-\frac{2}{\pi}\cot\left(\frac{\pi}{2}s\right) = t + C,$$
$$t = 0, s = 1 \Rightarrow C = 0$$
$$s = \frac{3}{2} \Rightarrow -\frac{2}{\pi}\cot\frac{3\pi}{4} = t \rightarrow t = -\frac{2}{\pi}$$

28. In 1970, the earth's population was 3.5 billion. If the rate of increase is 2% per year, the year in which the population will reach 50 billion will be closest to which of the following years?

a) 2100 b) 2150 c) 2200 d) 2300

answer: a

solution:

$$\frac{dP}{dt} = 0.02P \rightarrow \text{separate\& integrate}$$

$$P = Ce^{0.02t} \quad t = 0 \Rightarrow C = 3.5\text{billion}$$

$$50 = 3.5\ e^{0.02t} \rightarrow t = \frac{\ln(14.286)}{0.02} \approx 133\ years$$

$$1970 + 133 = 2103$$

29. Use Euler's method and 4 steps with $\Delta x = 0.1$ for the differential equation $y' = 2y$ to find an approximation for y, when $y(0) = 1$ and $x = 0.4$.

a) 1.452 b) 1.597 c) 1.872 d) 2.074

answer: d

solution:

X	Y	$\Delta x \bullet y' = \Delta y$
0	1	0.1(2)=0.2
0.1	1.2	0.1(1.2)(2)=0.24
0.2	1.44	0.1(1.44)(2)=0.288
0.3	1.728	0.1(1.728)(2)=0.3456
0.4	2.0736	

30. Which of the following differential equations is NOT logistic?

a) $P' = P - P^2$

b) $\frac{dy}{dt} = 0.01y(100 - y)$

c) $\frac{dx}{dt} = 0.8x - 0.004x^2$

d) $\frac{dR}{dt} = 0.16(350 - R)$

answer: d

solution: all others are of the form $y' = ry(C - y)$