Mu Alpha Theta National Convention: Denver, 2001

Sequences & Series Topic Test Solutions – Theta Division

1. $a_n - a_{n-1} = d$ $\quad 7-2=\underline{5}$ $\quad$ E

2. $\frac{a_n}{a_{n-1}} = r$ $\quad \frac{6}{3} = \underline{2}$ $\quad$ A

3. $\sum_{n=1}^{K}(2n-1) = K^2$ $\quad 28^2 = \underline{784}$ $\quad$ C

4. $\sum_{n=1}^{K} 2n = K^2+K$ $\quad 30^2+30 = \underline{930}$ $\quad$ D

5. $\sum_{n=1}^{K} n^2 = \frac{K(K+1)(2K+1)}{6}$ $\quad \frac{6}{6}\cdot 7\cdot 13 = \underline{91}$ $\quad$ D

6.

n	0	1	2	3
$2^{2^n}+1$	3	5	17	257

$3+5+17+257 = \underline{282}$ $\quad$ B

7. $\sqrt{6+x} = x$

$x^2 = 6+x$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$\underline{x=3}$

(x has to be >0) $\quad$ A

8. $a_2 = 4\cdot 2+7(-1)^2 = 8+7 = 15$

$a_4 = 4\cdot 4+7(-1)^4 = 16+7 = 23$

$15+23 = \underline{38}$ $\quad$ C

9. $\underbrace{3+4+\ldots+10}_{8 \text{ levels}}$ $\quad \frac{(3+10)\cdot 8}{2} = 52$, but these are ½ the cans $52\cdot 2 = \underline{104}$ $\quad$ D

10. $\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$ $\quad \therefore \quad \frac{K^2(K^2+1)}{2} = \underline{\frac{K^4+K^2}{2}}$ $\quad$ D

11. $\frac{a_n - a_1}{d} + 1 = n$ $\quad \frac{21-5}{d} + 1 = 5$

$\frac{16}{d} = 4 \quad \therefore d = 4$

$\underline{(9, 13, 17)}$ $\quad$ C

12. year 1 = 80000

year 15 = $80000 + 14\cdot 1500 = \underline{101000}$ $\quad$ D

13. $(2a_1+49d)\frac{50}{2} = 4025$

$2a_1 + 49d = 161$

$49d = 161-14$

$d = 3$

$\cancel{\ldots} =$

$a_{49} = 7+48\cdot 3 = 151$ $\quad$ C

14. $a_1 + a_1 r + a_1 r^2 = 147$

$a_1^3 r^3 = 21952$ $\quad \therefore \quad a_1 r = 28 \rightarrow a_1 + a_1 r^2 = 119$

$(a_1(1+r^2))^2 =$

$a_1^2 + a_1^2 r^2 + a_1^2 r^4 = ?$

$a_1^2(1+2r^2+r^4) = 119^2$

$a_1^2 + a_1^2 r^2 + a_1^2 r^4 = 119^2 - 28^2 = \underline{13377}$ $\quad$ B

15. $a_1 + (a_1+d_1) + \ldots + a_n \rightarrow m$ terms

$a_1 + (a_1+d_2) + \ldots + a_n \rightarrow n$ terms

$\frac{(a_1+a_n)m}{2} - \frac{(a_1+a_n)n}{2} = a_1 + a_n$

$\frac{m}{2} - \frac{n}{2} = 1$

$m - n = \underline{2}$ $\quad$ B

16. $a_1 = 3x$, $a_2 = 6x+1$, $a_3 = 9x+2$

$6x+1-3x = \cancel{\ldots}\ d$

$d = 3x+1$ $\quad a_{51} = a_1 + (50)\cdot d$

$3x + 150x + 50 = \underline{153x+50}$ $\quad$ A

17. $4, 4\cdot 2, \ldots, 4\cdot 2^5$ total $= \frac{4(2^6-1)}{2-1} = \cancel{\ldots}$

$63\cdot 4 =$

$\underline{252}$ $\quad$ A

18. $a_{10} = a_1 + 9d$

$a_{20} = a_1 + 19d$

$a_{40} = a_1 + 39d$

$a_{20} - a_{10} = 10d$

$a_{20} + 2(a_{20} - a_{10}) = a_{40}$

$12 + 2(12-27) = -\underline{18}$ $\quad$ B

19. $r = 1/3$ $\quad 2\cdot 3, 2\cdot 3^0, 2\cdot 3^{-1}, \ldots, 2\cdot 3^{-18}$

$\underline{20}$ terms $\quad$ A

20. $a_1 = a_0 + 1^2 = 1$

$a_2 = 1^2+2^2 \quad \ldots \quad a_3 = 1^2+2^2+3^2$, etc. $\quad$ D

$a_n = \frac{n(n+1)(2n+1)}{6}$ $\quad a_{20} = \frac{20(21)(41)}{6}$

$= \underline{2870}$

21. $(2a+9b)5 = 1150$

$2a + 9b = 230$

$(2a+19b)10 - 1150 = 3550$

$2a + 19b = 470$

$10b = 240$

$b = 24$

$a = 7$

$\sqrt{7^2+24^2} = \underline{25}$ $\quad$ C

22. $a_5 = a_1 + 4d = 4 \quad a_x = a_1 + (x-1)d = 104$

$a_1 = 4 - 4d \longrightarrow a_x = 4 - 4d + (x-1)d$

$a_x = 4 + (x-5)d = 104$

$(x-5)d = 100$

Find two numbers $(x-5)$ & d to multiply to get 100... since there are $\{2^2 \cdot 5^2 \ldots 3 \cdot 3 = 9\}$ 9 factors of 100, there are 9 possible values of x

D

23. $1{,}000{,}000 \cdot (150\%)^4$ = salary in 5th year

$1.5^4 \times 10^6 = 50625 \times 10^2$

C

24. $\underline{x}\ \underline{10} \cdot \underline{10} \cdot \underline{1}\ \underline{x}$ 100 ways to arrange first (and last digit) which can be 1–9

$[(1+2+\ldots+9)10^4 + (1+2+\ldots+9)]100 = 4.50045 \times 10^7$

$\underline{9}\ \underline{x} \cdot \underline{10} \cdot \underline{x} \cdot \underline{1}$ 90 ways to arrange 2nd (& 4th digit), 1–9

$[(1+2+\ldots+9)10^3 + (1+2+\ldots+9)10^1] \cdot 90 = 4.0905 \times 10^6$

$\underline{9} \cdot \underline{10} \cdot \underline{x} \cdot \underline{1} \cdot \underline{1}$ 90 ways to arrange 3rd digit, 1–9

$90[1+2+\ldots+9]10^2 = 4.05 \times 10^5$

$4.50045 \times 10^7 + 4.0905 \times 10^6 + 4.05 \times 10^5 = 4.95 \times 10^7$

D

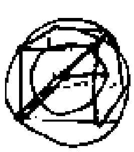
25. $.3\overline{12}_5 = \frac{3}{5} + \left(\frac{1}{25} + \frac{2}{125}\right) + \left(\frac{1}{5^4} + \frac{2}{5^5}\right) + \ldots$

$\frac{3}{5} + \frac{1/25}{1 - 1/25} + \frac{2/125}{1 - 1/25} = \frac{3}{5} + \frac{7/125}{24/25} =$

$\frac{3}{5} + \frac{7}{24} \cdot \frac{1}{5} = \underline{73/120}$

D

26.



r of first sphere 4

e of first cube $8/\sqrt{3}$

r of 2nd sphere $4/\sqrt{3}$

e of 2nd cube $8/\sqrt{3}^2$

⋮

r of nth sphere $= \frac{4}{\sqrt{3}^{n-1}} = \frac{4}{\sqrt{3}^{12}} = \frac{4}{3^6}$

$\rightarrow SA = \frac{4}{\bullet}\left(\frac{4}{3^6}\right)^2 \cdot \pi$

$= \frac{2^6 \pi}{3^{12}}$

A

27. sum of all integers between 100 – 2001

$\frac{1901(2100)}{2} = 1996050$

sum of multiples of 3

$\frac{(102 + 1998)633}{2} = 664650$

$1996050 - 664650 = \underline{1331400}$

C

28.

$2 + \left(\frac{3/2}{1 - 3/4}\right)2 \leftarrow$ on the way up & down

$2 + 12 = \underline{14\ m}$

A

29. logically, the minimum possible value should be achieved when d is as small as possible (nearly 0), making $a_1 = a_2 = \ldots$

so $\frac{a}{a+a} + \frac{a+a}{a+a} + \frac{a+a}{a} = \frac{1}{2} + 1 + 2 = \underline{7/2}$

B

30. $\sum_{n=1}^{\infty} \frac{4n}{5^n} = \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \frac{4}{5^4} + \ldots = \frac{4/5}{1 - 1/5}$

$+ \frac{4}{25} + \frac{4}{125} + \frac{4}{5^4} + \ldots = \frac{4/25}{1 - 1/5}$

$+ \frac{4}{125} + \frac{4}{5^4} + \ldots = \frac{4/125}{1 - 1/5}$

$+ \ldots$

$= \frac{4/5 + 4/25 + 4/125}{1 - 1/5}$

$\frac{4/5}{(1 - 1/5)^2}$

$= \frac{4/5}{(4/5)^2} = \underline{5/4}$

E

31.

$r_1 = 1,\ r_2 = \pi \cdot 1^2,\ r_3 = \pi(\pi \cdot 1^2)^2 = \pi^3$

$r_4 = \pi(\pi^3)^2 = \pi^7 \ \ldots \ r_n = \pi^{(2^{n-1} - 1)}$

$r_{24} = \pi^{2^{23} - 1} \quad 2\pi \cdot r_{24} = \underline{\pi^{2^{23}}}$

D

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32. $P(p)$ = prob Patrician wins = $1/6$
$P(s)$ = prob. Sean wins = $1/6$

prob she will win

$P(p) + (1-P(p))(1-P(s))P(p) + (1-P(p))(1-P(s))(1-P(p))(1-P(s))P(p) + \ldots$

$P(p)\left[1 + (1-P(p))(1-P(s)) + (1-P(p))^2(1-P(s))^2 + \ldots\right]$

$= \frac{1}{6}\left(\frac{1}{1-\left(\frac{5}{6}\right)^2}\right) = \frac{1/6}{11/36} = \underline{6/11}$ C

33. $\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$

$1 = A(x+3) + B(x-3)$

let $x=3$ ... $A = 1/6$
let $x=-3$ ... $B = -1/6$

$\sum_{n=4}^{\infty} \frac{1}{n^2-9} = \frac{1}{6}\sum_{n=4}^{\infty}\left(\frac{1}{n-3} - \frac{1}{n+3}\right)$

telescoping →

$= (1 - \cancel{\tfrac{1}{7}}) + (\tfrac{1}{2} - \cancel{\tfrac{1}{8}}) + (\tfrac{1}{3} - \cancel{\tfrac{1}{9}}) + (\tfrac{1}{4} - \cancel{\tfrac{1}{10}}) + (\tfrac{1}{5} - \cancel{\tfrac{1}{11}}) + (\tfrac{1}{6} - \cancel{\tfrac{1}{12}}) + (\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{13}}) + (\cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{14}}) + \ldots$

$\frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$

$= \frac{1}{6}\left(\frac{49}{20}\right) = \underline{\frac{49}{120}}$ B

34. $\frac{24}{5}, \frac{24}{6}, \frac{24}{7}, \frac{24}{8} \leftarrow = \underline{3}$ B

35. $2^n \bmod 7$ ... $2 \bmod 7 = 2$,
$2^2 \bmod 7 = 4$, $2^3 \bmod 7 = 1$, $2^4 \bmod 7 = 2$, etc.

$2^{3K} - 1$ is divisible by 7 D

$\sum_{K=1}^{100} 3K = \frac{3 \cdot 100 \cdot 101}{2} = \underline{15150}$

36. $(110-112) + (114-116) + \ldots + (1002-1004) + (1006-1008)$

$= (-2) + (-2) + \ldots + (-2) + (-2)$

$\frac{1006-110}{4} + 1 = 225$ terms, so $225 \cdot (-2)$
$= \underline{-450}$

E

37. list the last digit of each term (remainder when divided by 10)

n	0	1	2	3	4	5	6	7	8	9
a_n	2	5	(3)	5	7	5	3	5	7	5

pattern has 4 terms

C

a_{500} corresponds to $500 \bmod 4 = 4^{th}$ or 0^{th} term in pattern

$a_4 = \underline{(7)}$ ←

38. $S_4 - S_3 = a_4$

$\left(5 \cdot 4 - \frac{2}{4}\right) - \left(5 \cdot 3 - \frac{2}{3}\right) = 5 + \frac{2}{3} - \frac{1}{2} = \underline{31/6}$

C

39. test cases: $\frac{\binom{1}{0}}{4^0} + \frac{\binom{1}{1}}{4^1} = 1 + \frac{1}{4} = \frac{5}{4}$ $\quad \frac{\binom{2}{0}}{4^0} + \frac{\binom{2}{1}}{4^1} + \frac{\binom{2}{2}}{4^2} =$ $1 + 1/2 + 1/16 = 25/16$

$\frac{\binom{3}{0}}{4^0} + \frac{\binom{3}{1}}{4^1} + \frac{\binom{3}{2}}{4^2} + \frac{\binom{3}{3}}{4^3} = 1 + \frac{3}{4} + \frac{3}{16} + \frac{1}{64}$

$= \frac{125}{64}$ $\quad \ldots \frac{\binom{n}{0}}{4^0} + \frac{\binom{n}{1}}{4^1} + \ldots + \frac{\binom{n}{n}}{4^n} = \frac{5^n}{4^n}$

$1 + \frac{\binom{50}{1}}{4} + \frac{\binom{50}{2}}{4^2} + \ldots + \frac{\binom{50}{50}}{4^{50}} = \underline{\frac{5^{50}}{4^{50}}}$ C

40. $\sum_{n=5}^{\infty}\left(\log_2(n-1) - \log_2 n\right)$

$= (\log_2 4 - \cancel{\log_2 5}) + (\cancel{\log_2 5} - \cancel{\log_2 6}) + (\cancel{\log_2 6} - \log_2 7) + \ldots$

$= \log_2 4 = 2$ D