Mu Alpha Theta National Convention: Denver, 2001

Sequences & Series Topic Test Solutions – Alpha Division

1. $r=\frac{a_{n+1}}{a_n}$ $\quad 7/1=\underline{7}$ (A)

2. $d=a_{n+1}-a_n$ $\quad 12-4=\underline{8}$ (C)

3. $\sum_{k=1}^{n} 2k = n(n+1)$
$50\cdot 51=\underline{2550}$ (B)

4. $\sum_{k=1}^{n}(2k-1)=n^2$
$48^2=\underline{2304}$ (C)

5. $1+9+25+49=84$
$-(4+16+36+64)=-120$
$84-120=\underline{-36}$ (B)

6. $C_1=\frac{1}{2}\binom{2}{1}=1$ $\quad C_2=\frac{1}{3}\binom{4}{2}=2$
$C_3=\frac{1}{4}\binom{6}{3}=5$ $\quad 1\cdot 2\cdot 5=\underline{10}$ (D)

7. $\sqrt{20+x}=x$
$20+x=x^2$
$x^2-x-20=0$
$x=\frac{1+\sqrt{81}}{2}=\underline{5}$
(can't be <0) (D)

8. $a_3=18-5=13$ $\quad 13+37=50$
$a_7=42-5=37$ (B)

9. $\underbrace{2+3+\dots+11}_{10 \text{ levels}}=13\cdot 5=65$
this is ½ the stack
$65\cdot 2=\underline{130}$ (C)

10. $\sum_{n=1}^{x} n^2=\frac{x(x+1)(2x+1)}{6}$
let $x=\sqrt{k}$ ... $\frac{\sqrt{k}(\sqrt{k}+1)(2\sqrt{k}+1)}{6}$ (D)

11.

n	3^{n+1}	n^4
1	9	1
2	27	16
3	81	81
4	243	256
5	729	625
⋮	⋮	⋮

$3^{n+1}>n^4$
for all $\underline{n\geq 1}$
$\underline{\text{except } n=3,\ n=4}$
(B) (E)

12. $\sum_{k=1}^{3}\left(\sum_{n=1}^{5} n^k\right)=\sum_{k=1}^{3}(1+2^k+3^k+4^k+5^k)$
$=3+(2+4+8)+(3+9+27)+(4+16+64)$
$+(5+25+125)=3+14+39+84+155$
$=\underline{295}$ (B)

13. $\underbrace{10+12+\dots+34}_{13 \text{ terms}}=\frac{44\cdot 13}{2}=\underline{286}$ (D)

14. $T_n=\frac{n(n+1)}{2}=\frac{n^2}{2}+\frac{n}{2}$ $\quad S_n=n^2$
$2\left(\frac{n^2}{2}+\frac{n}{2}\right)-n=n^2$
$\underline{2T_n-n}$ (C)

15. $\frac{a_n-a_1}{d}+1=n$ $\quad \frac{100-13}{3}+1=30$ terms altogether
$30-2$ must be inserted $=\underline{28}$ (A)

16. 49 owls... $49\cdot 7=343$ mice killed
$343\cdot 7=2401$ ears of corn would have been eaten $\quad 2401\cdot 7=\underline{16807}$ lbs. grain (D)

17. $\sum_{i=0}^{500} 7^{2i}=\frac{(7^2)^{501}-1}{7^2-1}=\frac{7^{1002}-1}{48}$
$\underline{a=1002,\ b=+1,\ c=48}$ (D)

18. $a_1=x+1$ $\quad a_2=2y$ $\quad a_3=3x+y$ $\quad a_4=12$
$2y-(x+1)=3x+y-2y$ $\quad 12-3x-y=3x+y-2y$
$(a_2-a_1)=(a_3-a_2)$ $\quad (a_4-a_3)=(a_3-a_2)$
$2y-x-1=3x+y-2y$ $\quad 6x=12$
$4x-3y=-1$ $\quad x=2$
$y=3$
$a_1=3,\ d=3,\ n=14$
$\frac{(2\cdot 3+13\cdot 3)14}{2}=45\cdot 7=\underline{315}$ (C)

19. 
$\frac{\frac{5}{8}20}{1-5/8}$ up $+\frac{\frac{5}{8}20}{1-5/8}$ down $+20$
$=\frac{2\cdot 5\cdot 20}{3}+20=\frac{260}{3}$ (D)

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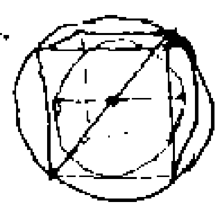
20. the sum is $(x-5)^2+\ldots+7^2$ E
start from 7^2 and move back...
$49+36=85+25=110+16=126+9=135+4=139+1$
$1^2+\ldots+7^2\ldots$ $=140$
so $\underline{x=5}$ also... x could = 8
$\underline{x=8}$ since $0+140=140$

21. $\begin{bmatrix} 1+\frac{1}{2}+\frac{1}{4}+\ldots & \frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\ldots \\ \frac{1}{3}+\frac{1}{12}+\frac{1}{48}+\ldots & \frac{1}{4}+\frac{1}{20}+\frac{1}{100}+\ldots \end{bmatrix}$ D
$= \begin{bmatrix} \frac{1}{1-1/2} & \frac{1/2}{1-1/3} \\ \frac{1/3}{1-1/4} & \frac{1/4}{1-1/5} \end{bmatrix} = \begin{bmatrix} 2 & 3/4 \\ 4/9 & 5/16 \end{bmatrix}$

22. $a_5=4$, $a_x=a_1+(x-1)d=2504$
$a_5=a_1+4d$ $\to$ $a_x=a_5+(x-5)d$
$a_1=a_5-4d$ $=4+(x-5)d=2504$
$(x-5)d=2500$
two numbers that multiply to get 2500: A

1, 2500	10, 250	there are $\underline{15}$ possible
2, 1250	25, 100	values for x that can
4, 625	20, 125	make this eqn. true
5, 500	50, 50	

23. 
1^{st} sphere, $r=9$
1^{st} cube, $e=\frac{2\cdot 9}{\sqrt{3}}=18/\sqrt{3}$
2^{nd} sphere, $r=\frac{18}{2\sqrt{3}}=9/\sqrt{3}$
2^{nd} cube, $e=\frac{2\cdot 9}{\sqrt{3}}/\sqrt{3}\cdot 6$
B n^{th} sphere, $r=\frac{9}{(\sqrt{3})^{n-1}}=\frac{9}{3^5}$ for $n=11$
$4\pi(9/3^5)^2=4\pi(1/3^3)^2=\frac{4\pi}{3^6}=\underline{\frac{4\pi}{729}}$

24. $\underline{\times}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{1}\ \underline{1}\ \underline{\times}$ 100 arrangements for 1^{st} & 7^{th} digits (1-9)
$100[(1+\ldots+9)(10^6+1)]=4.5000045\times10^9$
$\underline{9}\ \underline{\times}\ \underline{10}\ \underline{10}\ \underline{1}\ \underline{\times}\ \underline{1}$ 900 arrangements for 2^{nd} & 6^{th} digit (1-9)
$900[(1+\ldots+9)(10^5+10^1)]=4.050405\times10^9$
$\underline{9}\ \underline{10}\ \underline{\times}\ \underline{10}\ \underline{\times}\ \underline{1}\ \underline{1}$ 900 arrangements for 3^{rd} & 5^{th} digit (1-9)
$900[(1+\ldots+9)(10^4+10^2)]=4.0905\times10^9$
$\underline{9}\ \underline{10}\ \underline{10}\ \underline{\times}\ \underline{1}\ \underline{1}\ \underline{1}$ 900 arrangements for 4^{th} digit
$900[(1+\ldots+9)(10^3)]=4.05\times10^7$ A
sum $=\underline{4.95\times10^{10}}$

25. $\frac{\log(2n+2)}{\log(2n+1)}-\frac{\log(2n+4)}{\log(2n+3)}$
$=\left(\frac{\log 6}{\log 5}-\frac{\log 8}{\log 7}\right)+\left(\frac{\log 8}{\log 7}-\frac{\log 10}{\log 9}\right)+\left(\frac{\log 10}{\log 9}-\frac{\log 12}{\log 11}\right)+\ldots=\frac{\log 6}{\log 5}=\underline{\frac{\ln 6}{\ln 5}}$ A

26. $\left((-1)^0+(-1)^0\right)+\left((-1)^1+(-1)^1\right)+\ldots$
$=(1+1)+(-1-1)+(1-1)+(-1+1)+(1+1)+(-1-1)+\ldots$
$=2+(-2)+(0)+(0)+\ldots$
~~2002 terms~~ 244 terms..., $\frac{244}{4}=R\ 0$
0 terms left over after grouping in 4's B
so sum will be $\underline{0}$

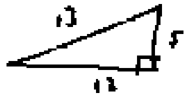
27. $r_1=1$, $r_2=\pi$, $r_3=\pi\cdot\pi^2=\pi^3$, $r_4=\pi\cdot\pi^6=\pi^7$
$r_n=\pi^{2^{n-1}-1}$ $r_{12}=\pi^{2^{11}-1}$ $A_{12}=r_{13}=\pi^{2^{12}-1}$
B $\underline{\pi^{4095}}$

28. $e^{\frac{2\pi k+\theta}{n}i}$ for values of K ranging from 0 to n−1 are n^{th} roots of $e^{i\theta}$...
C here, $\theta=0$, so $e^{i0}=1$
we want sum of n^{th} roots of $\underline{1}$
$x^n=1$ sum of roots $=\underline{0}$ for $n>1$!

29. we will want $a_1, a_2, a_3, \ldots$ as close together as possible so that they are nearly equal. let $a_1=a_2=\ldots=a_4=a$ D
$\frac{a}{a+a}+\frac{a+a}{a+a}+\frac{a+a}{a}=1+1/2+2=\underline{7/2}$

30. for $N<3$, $\lfloor\log_3 N\rfloor=0$
for $N<3^2$, $\lfloor\log_3 N\rfloor=1$
for $N<27$, $\lfloor\log_3 N\rfloor=2$
for $N<3^4$, $\lfloor\log_3 N\rfloor=3$
for $N<3^5$, $\lfloor\log_3 N\rfloor=4$
for $N<3^6$ $\lfloor\log_3 N\rfloor=5$
(729)
$\lfloor\log_3 729\rfloor=6$
$(9-3)1$
$+(27-9)2$
$+(81-27)3$
$+(243-81)4$
$+(729-243)5$
$+6$
$=\underline{3288}$ C

31. Assume a 5-12-13 rt. triangle



Perim = 30
Area = 30

D

altitudes = $5, 12, 60/13$

harm mean = $\frac{3}{1/5+1/12+13/60} = \frac{3}{(30/60)} = 6$

32. $j(j+1)(j+2) = \binom{j+2}{3}\cdot 6$

$6\sum_{j=1}^{999}\binom{j+2}{3}$ .. $\binom{3}{3}+\binom{4}{3}=1+4=5=\binom{5}{4}$

$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}=1+4+10=15=\binom{6}{4}$

$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\binom{6}{3}=15+20=35=\binom{7}{4}$

etc.

$6\sum_{j=1}^{999}\binom{j+2}{3} = \underline{\binom{1002}{4}\cdot 6}$

C

33. $(1+2+\ldots+n)^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

$1^3+2^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$ $\underline{y=3}$

C

34. a_{15} should equal the average of the 1st 29 terms: $\frac{a_1+a_2+\ldots+a_{29}}{29} = a_{15}$

A

$\frac{a_{15}}{b_{15}} = \frac{a_1+a_2+\ldots+a_{29}}{b_1+b_2+\ldots+b_{29}}$ $\frac{S_a(29)}{S_b(29)} = \frac{154}{66} = \underline{\frac{7}{3}}$

35. determine last digit of each term
(this will be remainder when div by 10)

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	2	3	3	5	7	5	3	5	7	5	3	5

$\frac{1492}{4}$ has remainder 0, so we can use 4th # in pattern $\underline{7}$

B

36. The only way to avoid winning or losing is to roll a 1 or 4 each turn (with 1/3 prob.)

$\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\ldots$ this will become very close to $\underline{0}$

turn 1, 2, 3, ...

E

37. ~~$x+x^2+x^3+\ldots$~~

$x^2+x^3+\ldots+x^{11}+x^{12} = \frac{x^2(x^{11}-1)}{x-1}$

$x^3+\ldots+x^{11} = \frac{x^3(x^9-1)}{x-1}$

$x^4+\ldots+x^{10}$

$\vdots$

x^7

$$\frac{x^2(1-x^{11})+x^3(1-x^9)+x^4(1-x^7)+x^5(1-x^5)+x^6(1-x^3)+x^7(1-x)}{1-x}$$

$$\frac{(x^2+x^3+x^4+x^5+x^6+x^7)-(x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8)}{1-x}$$

$$\frac{(x^2-x^8)(1+x+x^2+x^3+x^4+x^5)}{(1-x)}\cdot\frac{(1-x)}{(1-x)} =$$

$$\frac{(x^2-x^8)(1-x^6)}{(1-x)^2} = \underline{\frac{x^2(1-x^6)^2}{(1-x)^2}}$$

D

38. $\frac{6-24n}{4n^4-4n^3+n^2} = \frac{A}{n}+\frac{B}{n^2}+\frac{C}{2n-1}+\frac{D}{(2n-1)^2}$

$6-24n = An(2n-1)^2+B(2n-1)^2+Cn^2(2n-1)+Dn^2$

let $n=0$ ... $B=6$ let $n=1/2$, $D=-24$

let $n=1$ ... let $n=-1$

$A+B+C+D=-18$ $-9A+9B-3C+D=30$

$A+C=0$ & $\Rightarrow$ $-9A-3C=0$

$A=0$ & $C=0$

$\frac{6}{n^2}-\frac{24}{(2n-1)^2} = 6\cdot\frac{\pi^2}{6}-24\cdot\frac{\pi^2}{8} = \pi^2-3\pi^2 = \underline{-2\pi^2}$

C

39. test cases: $\frac{\binom{1}{0}}{2^0}+\frac{\binom{1}{1}}{2^1} = \frac{3}{2}$ $\frac{\binom{2}{0}}{2^0}+\frac{\binom{2}{1}}{2^1}+\frac{\binom{2}{2}}{2^2} = \frac{9}{4}$

$\frac{\binom{3}{0}}{2^0}+\frac{\binom{3}{1}}{2^1}+\frac{\binom{3}{2}}{2^2}+\frac{\binom{3}{3}}{2^3} = \frac{27}{8} \Rightarrow \frac{\binom{n}{0}}{2^0}+\ldots+\frac{\binom{n}{n}}{2^n} = \underline{\frac{3^n}{2^n}}$

A

40.

40. Let the sum be S.

$$S < 21\cdot\frac{1}{10} + 23\cdot\frac{1}{11} + 25\cdot\frac{1}{12} + \cdots + 39\cdot\frac{1}{19} + \frac{1}{20} \approx 20.x$$

$$S > \frac{1}{10} + 21\cdot\frac{1}{11} + 23\cdot\frac{1}{12} + 25\cdot\frac{1}{13} + \cdots + 39\cdot\frac{1}{20} \approx 19.y$$

$\Rightarrow$ A or E

$$S \approx 11\cdot\frac{1}{10} + 23\cdot\frac{1}{11} + 25\cdot\frac{1}{12} + \cdots + 39\cdot\frac{1}{19} + 40\cdot\frac{1}{20} \approx 20.26 \quad \text{(E)}$$

Also, on TI-83+, sum(seq(A^(-.5), A, 100, 400) = 20.075...

That's so close to the line that you'd need a really good lower-bound approximation.