Mu Alpha Theta National Convention: Denver, 2001

Number Theory Topic Test Solutions – Alpha Division

1. **The easiest way to solve this problem is to find the total possible values of m because for each m there is exactly one possible n to pair with. The possible values of m consist of the total number of possible factors (positive and negative) of 120. The total number of positive factors of a number can be derived from that number's prime factorization. Add one to each of the exponents in the prime factorization (the total number of choices for the exponent of that prime in a factor) and multiply those numbers together (they are chosen independently from one another to identify each factor). The prime factorization of 120 has three primes (2, 3, 5) with exponents of 3, 1, and 1 respectively. So there are (3+1)(1+1)(1+1) = 16 possible positive values for m, and also 16 negative values. For each m, there is exactly one n. There are 32 different ordered pairs.**

2. **512 =** 2^9**.** $\sum_{n=0}^{9} 2^n$ **= (**2^{10}**-1)/(2-1) = 1023.**

3. **The trick is to find the number of powers of 10 which divide 634!. This means finding the exponents of 2 and 5 in the prime factorization of 634!. The exponent of 2 will clearly be greater than the exponent of 5, so we only need find the exponent of 5. 634/5 = 126 with a remainder. 126/5 = 25 with a remainder. 25/5 = 5 and 5/5 = 1. That means that there are 126 + 25 + 5 + 1 = 157 times in which 5 is included in the prime factorization of 634!. Thus 157 is the answer.**

4. **Using the same method of factor counting as in problem #1, we can arrange 32 as the product of exponents (+1) in several ways. 32 = (4)(2)(2)(2) = (4)(4)(2), etc. We can make the problem easier by noting that we can produce a smaller number with four factors using only powers of 2 than with a single factor of 2 and a prime greater than 2 x 2 (for instance, 8 is less than 10 or 14). Noting such relationships (we could test 3 to the third vs. 3 x 7 or 3 x 11) we can see that 32 = (4)(2)(2)(2) produces the smallest possible integer using the smallest primes (2, 3, 5, and 7). (2)(2)(2)(3)(5)(7) = 840.**

5. **A pair of twin primes can be written as (x – 1) and (x + 1). Their product is (** x^2 **- 1). For (** x^2 **- 1) > 6000, we must find x, such that** x^2 **> 6001. From this we know that we are looking for x > 77. The first pair of twin primes from there is 101 and 103. Their sum is 204.**

6. **There is a theorem that I have heard called the "Chicken Mcnugget Theorem" which gives a solution for a diophantine equation with only two relatively prime variables (a and b) instead of three. The formula is fairly easy to derive and shows that d = ab – a – b. I leave it as an exercise to the solvers to derive the formula for more variables.**

7. **4,199 = 13 x 17 x 19. 13 + 17 + 19 = 49.**

8. 432_8 **= 4(64) + 3(8) +2 = 282 =** n^2**+ 6n + 2. Thus** n^2**+ 6n – 280 = 0. Solutions for n are 14 and –20. Discard the negative solution.**

9. **33,331 is the only one of the potential answers with leaves a remainder of 4 when divided by 7. A quick way to solve would be to note that $31 \equiv 3 \pmod 7$ and that 331 = $10(31) + 21 \equiv 10(3) \pmod 7 \equiv 2 \pmod 7$ and working up through the possible answers quickly that way.**

10. **For the three integers a, b, and c. $abc \equiv (3)(3)(3) \pmod{13} \equiv 27 \pmod{13} \equiv 1 \pmod{13}$.**

11. **Using the same method of factor counting as in problem #4, we can arrange 48 as the product of exponents (+1) in several ways. 48 = (4)(3)(2)(2) = (6)(4)(2), etc. We can see that 48 = (4)(3)(2)(2) produces the smallest possible integer using the smallest primes (2, 3, 5, and 7). (2)(2)(2)(3)(3)(5)(7) = 2,520.**

12. **Call the number AB where A is the tens digit and B is the units digit.**

$$BA – AB = 10(B - A) + (A - B) = 9(B - A) = 18. \text{ Thus } B – A = 2.$$

13. 144_6 **= 36 + 24 + 4 = 64. It is easy to see that this is equal to** 100_8.

14. **324 =** $2^2 \times 3^4$. **Consider that the sum of all of the factors can be determined by:**

$(2^2 + 2^1 + 1)(3^4 + 3^3 + 3^2 + 3^1 + 1)$ because each and every factor is represented one and only once as one of the products of a power of each of the prime numbers in 324's prime factorization. Also notice that $(2^2 + 2^1 + 1)(2 - 1) = (2^3 - 1)$. So a more compact formula can be derived to find the sums of factors of ANY integer. [Try to derive this formula completely as an exercise.]

The sum of all of its positive factors is thus $(2^3 - 1)(3^5 - 1)/[(2 - 1)(3 - 1)] = 847$.

15. **$N \equiv -1 \pmod 3$ and $N \equiv -1 \pmod 4$, thus $N \equiv -1 \pmod{12}$ which implies $N \equiv -1 \pmod 6$. The remainder is therefore 5.**

16. **Given the formula for the sum of the first n counting numbers, we can relate:**

$n(n + 1) \equiv 0 \pmod{249}$ => $n(n + 1) \equiv 0 \pmod{83}$. We can use this as a limiting factor and then use $n(n + 1) \equiv 0 \pmod{83}$ to find that our smallest n is 83.

17. Factoring 1,288 shows a prime factorization including 2 to the third power, 7, and 23. Since one of the ages is a multiple of 23, only a couple of attempts at dividing 1,288 need be made to see that (23, 56) and (28, 46) are the only age possibilities with differences less than 40. Only the first one has a difference greater than 20. My father is 56 -- and will be until a few days after this competition, though I will have turned 24 a few weeks before. ;-D

18. This problem is much easier given a knowledge of the arithmetic mean-geometric mean (A.M.-G.M) inequality. For any two positive numbers a and b, their A.M. is greater than or equal to their G.M. Given a fixed G.M., the A.M. is smaller when the difference between a and b is smaller (this is left as an exercise for the students to prove). So, the solution to this problem involves finding three numbers that are relatively prime and considering whether or not there is a way to reduce the A.M. of any pair of them (given that they HAVE a G.M. already). The numbers turn out to be 3, 5, and 16. The sum is 24.

19. 280a ≡ 0 (mod 7), 280a – 60a ≡ -1 (mod 7) => 220a ≡ 6 (mod 7).

20. 4A4B ≡ 0 (mod 72), thus 4A4B ≡ 0 (mod 8) and 4A4B ≡ 0 (mod 9).

(1) 4A4B ≡ 0 (mod 8) => 4A4B – 4040 ≡ 0 (mod 8) => 100(A) + B ≡ 0 (mod 8).

(2) 4A4B ≡ 0 (mod 9) => A + B + 8 ≡ 0 (mod 9). Thus A + B must be 1 or 10.

From (1) we see that B must be even, so the solver need only plug in the small number of potential solutions to both the final equations from (1) and (2). The only solution is (2, 8) meaning that A = 2.

21. The difference between their ages must be a multiple of the LCM of Katie's ages on those 5 birthdays. The smallest that LCM could be is 60. On the last birthday Bart must have been 65.

22. 17x ≡ 177 (mod 1777) => 17x ≡ 177 + 3(1777) (mod 1777) => 17x ≡ 5,508 (mod 1777) => x ≡ 324 (mod 1777) and thus 177x ≡ 177(324) (mod 1777) ≡ 57,348 (mod 1777) ≡ 484 (mod 1777).

23. Let the two numbers be (s/2 + d/2) and (s/2 – d/2) where s is the sum of the numbers and d is their (positive) difference. Their product is thus ($s^2 - d^2$)/4. Thus we know that $s^2 - d^2$ = 5,412. Minimizing d means minimizing s^2. The smallest possible s^2 greater than 5,412 is the square of 74. Luckily, we need look no further as |s| = 74 gives an integer solution for d. Min(d) = 8.

24. $.\bar{1}$ **in base six is equivalent to the sum of the geometric series whose first term is 1/6 and whose common ration is also 1/6. The sum is 1/5 which is .2**

25. The sum of the first n perfect cubes is $[(n)(n+1)/2]^2$. **Plugging in yields (40)(41)/2 = 820. 820 x 820 = 672,400.**

26. A positive integer has an even number of (positive) factors if and only if it is a perfect square. This can be shown in several ways, the easiest of which is to point out that for any factor of a given integer, one can always find a distinct factor by dividing the factor into the original number *unless* the given factor *is* the square-root of the integer. The sum of the first 20 perfect squares is (20)(21)(41)/6 = 2,870.

27. An integer expressed in a base, B, is a multiple of (B – 1) if and only if the sum of the integer's digits is a multiple of (B – 1). This can be easily proven by induction (or other means) and is left as an exercise for the students. The sum of the digits of N is 30 and N is at least 10 and less than 20. (B-1) can only be 10 or 15. B = 11 is the only one of the answer choices which will satisfy.

28. First, we divide the total number of killed aliens by 9 (we will multiply back later) to get 281 kills. By the pigeonhole principle, Pat B. must have killed at least 8 aliens. Multiplied by 9 means he must have killed at least 72.

29. 7^4 **= 2,401 ≡ 400 + 1 (mod 1,000). Take that result to the fifth power:**

$7^{20} \equiv (400+1)^5$ **(mod 1,000) ≡ 5(400) + 1 (mod 1,000) ≡ 1 (mod 1,000).**

Thus $7^{707} \equiv 7^{20(35)}\, 7^4\, 7^3$ **(mod 1,000) ≡ (1)(401)(343) (mod 1,000) ≡ 543 (mod 1,000).**

Hence the hundreds digit of 7^{707} **is 5.**

30. Since x = 41m + 3, we can expand the polynomial of x^p **in terms of m and see that the factor of 41 is present in all terms except for the** 3^p **term. Then from Fermat's Little theorem, we can simplify the task by knowing that** $3^{40} \equiv$ **1 (mod 41). Finally, all we need to do is calculate** 3^6 **(mod 41). 729 ≡ 32 (mod 41).**

31. The problem is to find the sum of the first 25 perfect squares that are not perfect fourth powers. This means finding the sum of the first 30 perfect squares and subtracting out the five which are perfect fourth powers. The sum of the first 30 perfect squares is

(30)(31)(61)/6 = 9,455. 1 + 16 + 81 + 256 + 625 = 979. 9,455 – 979 = 8,476

32. From $x \equiv 2 \pmod 4$, we can say that x = 4a –2 for some positive integer, a. Then from the second equation, $x \equiv 3 \pmod 9$, we can say that $4a - 2 \equiv 3 \pmod 9$ which implies that $4a \equiv 5 \pmod 9$ => $4a \equiv -4 \pmod 9$ => $a \equiv -1 \pmod 9$. We can say that a = 9b –1 for some positive integer, b. Thus x = 36b – 6. Finally, from the last equation, we can substitute for x once again: $36b - 6 \equiv 7 \pmod{49}$ => $36b \equiv 13 \pmod{49}$ => $36b \equiv -36 \pmod{49}$ => $b \equiv -1 \pmod{49}$. So, for b we can say that b = 49c – 1 for some positive integer, c. Relating x to c yields that x = 1764c – 42. Plugging in 3 for c yields 5,250. Problems like this should not be thought of as difficult, they merely require a little algebra work.

33. 360 has 4 x 3 x 2 = 24 positive divisors. For each divisor d, there is exactly one other divisor m such that dm = 360. Thus there are 12 pairs of divisors, the product of each pair being 360. Thus the product of all the divisors is 360^{12}.

34. (A, B, C, D) = (A, B, 4B, 4B). 4B is a multiple of 11 meaning that B is a multiple of 11. Let B = 11x, x is an integer. Then A + B + C + D = A + 99x = 100. Since x is an integer, x can only be 1 and thus A = 1.

35. By Fermat's Little Theorem we know that $5^{13-1} \equiv 1 \pmod{13}$. From this we can determine that $5^{(12)(25)+1} \equiv (1^{25})(5) \pmod{13} \equiv 5 \pmod{13}$.

36. As in problem #27, the sum of the digits of the number must be equal to 5 when expressed in base 6. The smallest is $11{,}111_6$. The second smallest is $101{,}111_6$. The third smallest is $110{,}111_6$ = 7,776 + 1,296 + 36 + 6 + 1 = 9,115.

37. Solving this problem involves a degree of deduction taking several factors into consideration. We can rule out even values of *N*. We can also note that phi(10) = 4 and phi(100) = 40. This will help limit our search as we know that the units digit of 3^N repeats in a 4-cycle and the last pair of digits repeats in (at most) a 40-cycle. In fact, noting that 3 x 3 x 3 x 3 = 81 = (80 + 1), we can see by binomial expansion that taking 81 to the fifth power produces a number with a units digit of 1 and a tens digit of 0. Thus 3^N repeats its last two digits in a 20-cycle. Now we must simply look for where $3^N \equiv 0 \pmod{20}$ and adjust *N* by adding/subtracting multiples of 20. We thus need only check the first 20 positive integers (and only the 10 odd ones of those).

We can rule out most of these by comparing the 4-cycle of units digits. If $N \equiv 1 \pmod 4$, then the units digit of 3^N will be 3. If $N \equiv 3 \pmod 4$, the units digit will be 7. The only *N* that need be tested are thus 7 and 13. $3^N \equiv N \pmod{20}$ for 7, but not 13. The tens digit of 3^7 is 8, thus 87 is the only solution such that $3^N \equiv N \pmod{100}$.

38. This problem requires separating the cases of p =2 and p =3 from other cases because they are factors of 24. They clearly have their own solutions for b. For p > 3 we will evaluate: We can first write p as (2m + 1) and then evaluate p^2 (mod 8) .

p^2 **= 4** m^2 **+ 4m + 1 = 4m(m + 1) + 1. Either m or (m + 1) is even, thus**

4m(m+1) + 1 $\equiv$ 1 (mod 8). Also, such p are either congruent to 1 or –1 (mod 3) and thus p^2 **is congruent to 1 (mod 3).** p^2 **is thus also congruent to 1 (mod 24). This is true for all primes that are not 2 or 3 and so there are exactly 3 possible values for b.**

39. The Fibonacci numbers will always be cyclical in any mod because a term is defined by its predecessors and there are a limited number of possible combinations for a pair of predecessors which would then produce the same cyclical pattern each time that pair occurs. The trick is just to write down the modular residues of the Fibonacci numbers (mod 7) until the cycle is found: 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, etc. The 1st pair of terms (1, 1) reappeared as the 17th pair. The cycle is thus a 16-cycle, thus m = 16.

40. **The sum, A, need only be calculated up to the point at which all subsequent terms are multiples of 108. Thus only the first 8 terms need be calculated. The sum of those terms is 1 + 2 + 6 + 24 + 120 + 720 + 5,040 + 40,320 = 46,233. The remainder when 46,233 is divided by 108 is 9.**