Mu Alpha Theta National Convention: Denver, 2001
Proofs Test

1. a. Prove that the sum S of an infinite geometric series with first term a and common ratio r is $S = \frac{a}{1-r}$. (7 points)

   b. Given the result of part a, prove that the sum S of the first n terms of a geometric series with first term a and common ratio r is $S = \frac{a\left(1-r^n\right)}{1-r}$. (3 points)

2. Prove that the sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers. (10 points)

3. Prove that for a right triangle with legs of lengths a and b and a hypotenuse of length c, $a^2 + b^2 = c^2$. (10 points)

4. Let $\{a_n\}_{n>0}$ be an arithmetic sequence whose first term and common difference are both nonzero. Suppose that a_6, a_4, and a_{10} are three consecutive terms of a geometric sequence. If $S(n)$ equals the sum of the first n terms of a_n then

   a. Prove that $S(10) = 0$. (7 points)

   b. Show that $S(6) + S(12) = 0$. (3 points)

5. Prove that in a set containing 52 integers, there always exists a pair of integers such that their sum or difference is a multiple of 100. (10 points)

6. Given segments with length 1, a, and b, find a method to construct a segment of length ab using only a straightedge and compass, and prove that it works. (10 points)

7. a. Show that $\sqrt{2}$ is irrational. (3 points)

   b. Show that $2^{1/2^n}$ is irrational for all natural numbers n. (7 points)

8. Prove that between any two rational numbers, there is at least one irrational number. (10 points)

9. Two real numbers a and b have a product of 1. Prove that $a^6 + 4b^6 \geq 4$. (10 points)

10. Show that $\cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{9\pi}{5}\right) = \frac{1}{2}$. (10 points)

11. On a given circle, six distinct points, A, B, C, D, E, and F, are chosen at random. Prove that the probability triangles ABC and DEF do not overlap is equal to $\frac{3}{10}$. (10 points)

12. Prove that for every integer m there is at least one integer n such that $m + n + 1$ is a perfect square and $mn + 1$ is a perfect cube. (10 points)