Mu Alpha Theta National Convention: Denver 2001
Interschool Test – Solutions

1. Assume that person D is false. This implies that A and B are both true , which forces C to be true. But C's statement is clearly false, so we have a contradiction. Now suppose D true and C is false. This implies that A and B are both false. But then B claims that C and D are opposite types of people, which is true—again, a contradiction. If D and C are both true then it follows that A is true and B is false, which is consistent. Thus, every statement is true except for B.

2. We use the fact that $ABCDEF$ is an equiangular hexagon iff $AB + BC = DE + EF$ and $EF + FA = BC + CD$.

   A. If $AB = 1$ and $BC = x$ then $1 + x = 4 + 5$, making $x = 8$.

   B. Without loss of generality, let $BC = 6$. Consequently, the only possibilities for A are 1, 2, or 3 since if $A \geq 4$, there are no choices for $AB + BC = DE + EF$ to be true. Thus, we have the following cases (up to interchanging DE and EF):

$$\begin{aligned} AB + BC &= DE + EF \\ 1 + 6 &= 2 + 5 \\ 1 + 6 &= 3 + 4 \\ 2 + 6 &= 3 + 5 \\ 3 + 6 &= 4 + 5 \end{aligned}$$

   From these equations along with $BC + CD = EF + FA$, we find that the only possible (cyclic) orderings for the sides are $(1, 6, 2, 4, 3, 5)$ and $(1, 6, 3, 2, 5, 4)$.

3. If we let $(x_1, y_1, z_1) = (1, 1, 3)$, $(x_2, y_2, z_2) = (2, 2, 1)$, $(x_3, y_3, z_3) = (3, 4, 8)$, and $(x_4, y_4, z_4) = (5, 6, 9)$, then the equation of the sphere is given by

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 11 & 1 & 1 & 3 & 1 \\ 9 & 2 & 2 & 1 & 1 \\ 89 & 3 & 4 & 8 & 1 \\ 142 & 5 & 6 & 9 & 1 \end{vmatrix} = 0$$

   Evaluating this rather large determinant, we find that the equation of the sphere is $5x^2 + 5y^2 + 5z^2 - 207x + 103y - 57z + 220 = 0$. Completing the square yields a center of $(207/10, -103/10, 57/10)$.

4. Start at $(x, y) = (-1, -1)$ and generate the terms in a Pascal's Triangle-like manner. The answer is 58786.

5. *The Man Without A Face.*

6. The function $f(x)$ counts the number of ones in the binary expansion of x. Since $1 \leq 2^{10} - 1 \leq 2001$, the maximum value of f on the given interval is 10.

7. This value is Miller's Number, 7.

8. Let $x^2 = 6a + 4$ and $y^2 = 2a + 1$. A little algebra and we get the Pell equation $x^2 - 3y^2 = 1$; the smallest solution to this equation is $(2, 1)$. Recall that from an initial solution (x_0, y_0) of a Pell equation $x^2 - Dy^2 = 1$, another solution (x_n, y_n) can be obtained by computing $(x_0 + y_0\sqrt{D})^n$ namely, $x_n + y_n\sqrt{D} = (x_0 + y_0\sqrt{D})^n$. Raising $2 + \sqrt{3}$ to several powers, we get $(2 + \sqrt{3})^2 = 7 + 4\sqrt{3}$, $(2 + \sqrt{3})^3 = 26 + 15\sqrt{3}$, etc. The smallest odd value of y such that a is a natural number is 15, making $a = (15^2 - 1)/2 = 112$.

9. Given that A $= 10$, B $= 11$, $\ldots$, Z $= 35$, the sequence spells out MU ALPHA THETA when converted to base-36. Since $\text{ALPHA}_{36} = 10(36)^4 + 21(36)^3 + 25(36)^2 + 17(36) + 10 = 17808958$, that is the answer.

10. If there are $2i$ occurrences of R, these can occur in $\binom{8}{2i}$ ways and the remaining spots can be filled with Es and Ls in 2^{8-2i} ways. Summing over the possible values of i yields $\sum_{i=0}^{4} \binom{8}{2i} 2^{8-2i} = 3281$.

11. The next day could be the son's birthday, the father's birthday, or both (this doesn't work out). Let the father's age be $10T + U$, where $0 < T, U < 10$. Thus, the first case is $10T + U = 2(10U + T + 1)$ and the second case is $10T + U + 1 = 2(10U + T)$. The former gives the father-son age ordered pair $(52, 25)$ and the latter yields $(73, 37)$.

12. In $\triangle ABC$, let $\angle A = 3\angle B$. Let D be a point on BC such that $\angle ABC = \angle BAD$. This gives rise to isosceles triangles ABD and ADC. By Stewart's Theorem

$$c^2 b + b^2(a - b) = a(a - b)^2 + ab(a - b) \rightarrow (a^2 - b^2)(a - b) = bc^2$$

Use the above result along with the Triangle Inequality and a little trial and error to generate the triangle with sides $(a, b, c) = (10, 8, 3)$ having a minimal perimeter of 21.

13. A representation for each part is listed below.

    A. $(9 - 6) \times 4 \times 2$

    B. $(4 + 1 + 3) \times 3$

    C. $7 \times 3 + 2 + 1$

    D. $8 \times (4 - 7/7)$

    E. $(9 - 5) \times 6 \times 1$

    F. $6 \times (7 + 1 - 4)$

    G. $8 \times (7 - 5) + 8$

    H. $(8 - 5) \times 6 + 6$

I. $2 \times (8+5) - 2$

J. $4 \times (3+4) - 4$

K. $7 \times 5 - 9 - 2$

L. $(5 - 1/5) \times 5$

M. $6/(1 - 6/8)$

N. $8/(3 - 8/3)$

14. The numbers may be (uniquely) arranged in cyclic order as follows:

$$1, 8, 28, 21, 4, 32, 17, 19, 30, 6, 3, 13, 12, 24, 25, 11,$$
$$5, 31, 18, 7, 29, 20, 16, 9, 27, 22, 14, 2, 23, 26, 10, 15, 1$$

15. Use a computer program to simplify the work of number-hunting. The unique cycle is 9261 (cube), 6133 (prime), 3364 (square), 6441 (triangular), 4181 (Fibonacci), and 8192 (power of 2).

16. Unfold the cube so that it makes a perfectly symmetric "cross" with isosceles right triangles at each end (these are the four triangles formed when the top face is cut by its diagonals). We can then place this figure on top of a square whose side length is equal to the distance between two adjacent corners. This distance is equal to $\sqrt{2}/2 + \sqrt{2} + \sqrt{2}/2 = 2\sqrt{2}$ so the area of the wrapping paper is $(2\sqrt{2})^2 = 8$.

17. Make a triangular grid with all the possible products for Hermione, and the same for Ron's sums. In the "first round," Hermione doesn't know the numbers, so her product wasn't 1, 2, 3, 5, 7, 10, 14, 15, 21, 27, 20, 28, 32, 25, 30, 35, 40, 45, 42, 48, 54, 49, 56, 63, 64, 72, or 81 (or she'd have known the two digits, as each of these numbers appears only once in her grid). Cross all those numbers out in her grid, and their corresponding positions in Ron's grid (since he knows that those digit combinations are impossible or Hermione would have known the answer). Now, Ron doesn't know the numbers either, so his sum can't be 4, 12, or 13, so cross out these positions in both grids. Now, Hermione doesn't know the numbers, so her product isn't 4. Now, Ron crosses out 5. Hermione crosses out 6, Ron kills 7, and Hermione knows the answer. So her product must have been 12 and the digits are 2 and 6.

18. The corresponding system of congruences is

$$\begin{aligned} x &\equiv 11 \bmod 21 \\ x &\equiv 2 \bmod 37 \\ x &\equiv 33 \bmod 51 \end{aligned}$$

Substitute $x = 21a + 11$ into the middle congruence to obtain $a \equiv 26 \bmod 37$, or $a = 37b + 26$ for some integer b. Thus $x = 21(37b+26)+11 = 777b+557$. Plugging this into the bottom congruence, we get $b \equiv 4 \bmod 51$; this means $b = 51c + 4$. Therefore, $x = 777(51c+4)+557 = 39627c+3665$. The smallest solution to the system in natural numbers occurs when $c = 0$ namely, when $x = 3665$.

19. Clearly, Julia is younger than John. Let U be Julia's age, and O be John's age. Julia will be John's age in $O - U$ years, at which point John will be $2O - U$ years old. Six less than that is $2O - U - 6$. One quarter of that is $(2O - U - 6)/4$. It's implied that this is less than Julia's present age, so we'll say that she was that old $U - (2O - U - 6)/4$ years ago, which "simplifies" to $(5U - 2O + 6)/4$ years ago. That long ago, John was $O - (5U - 2O + 6)/4 = (6O - 5U - 6)/4$, which is Julia's present age: $U = (6O - 5U - 6)/4$. Thus, we have $3U + 2 = 2O$. Assuming the sum is a two-digit number, U could be as much as 40 and O could be as much as 60. The acceptable (U, O) ordered pairs are (2, 4), (4, 7), (6, 10), (8, 13), (10, 16), (12, 19), (14, 22), (16, 25), (18, 28), (20, 31), (22, 34), (24, 37), (26, 40), (28, 43), (30, 46), (32, 49), (34, 52), (36, 55), and (38, 58). Of these, only (34, 52) works, i.e. the digits of $U + O$ reversed is equal to $2U$. Thus, the absolute age difference is 18.

20. Proceed backwards. There is no offer that Pirate 2 can make that Pirate 1 will vote for so Pirate 2 will vote for any offer Pirate 3 makes so he can live. Thus, Pirate 3 will suggest he gets all 100 Galleons. Pirate 4 can offer Pirates 1 and 2 one coin each, keeping 98, and get them to vote with him, since he would have received nothing from Pirate 3. Pirate 5 can offer 1 coin to Pirate 3, and 2 coins to either Pirate 1 or 2, keeping 97 for himself. Thus, (A, B, C, D, E) can equal $(2, 0, 1, 0, 97)$ or $(0, 2, 1, 0, 97)$. Either way, the sum of their squares is 9414.

21. Have people with "similar" speeds cross the bridge together. First, have Harry and Hermione cross, with Harry returning the flashlight. Then, have Ron and Neville cross and make Herm bring back the flashlight. Finally, have Harry and Herm cross once more for a total time of $2 + 1 + 10 + 2 + 2 = 17$ minutes.

22. Use trial-and-error. A possible coloring is shown below.

G	W	B	Y
B	Y	R	G
R	G	W	B
W	B	Y	R

23. For 192 to be the difference between squares, it must be possible to express it as the sum of consecutive odd numbers. For a number to be expressible as the sum of consecutive odd numbers, it must be possible to decompose it into an odd number times an odd number (one being the central term of an arithmetic sequence, the other being the number of terms) or an even number times an even number (one being the mythical "central" term of an arithmetic sequence, the other being the number of terms of that sequence). Now 192 is clearly not of the first type (being even), but is equal to

$2 \times 2 \times 48$, so is of the latter type.

$$192 = (2 \times 1)(2 \times 48) \to 95 + 97 \to 2209, 2401$$
$$192 = (2 \times 2)(2 \times 24) \to 45 + 47 + 49 + 51 \to 484, 676$$
$$192 = (2 \times 3)(2 \times 16) \to 27 + 29 + \cdots + 37 \to 169, 361$$
$$192 = (2 \times 4)(2 \times 12) \to 17 + 19 + \cdots + 31 \to 64, 256$$
$$192 = (2 \times 6)(2 \times 8) \to 5 + 7 + \cdots + 27 \to 4, 196$$

Thus, $2401 - 4 = 2397$.

24. If A were at Z, the furthest point, B, would be U, and the distance would be $10\sqrt{5}$ whether the ant crossed $STYZ$ and $TUXY$, $STYZ$ and $STUV$, $SVWZ$ and $STUV$, $SVWZ$ and $UVWX$, $WXYZ$ and $TUXY$, or $WXYZ$ and $UVWX$. As A moves towards Y, it's clearly getting closer to points on UX and UT by the $STYZ$ and $TUXY$ route, so the new furthest point is either U or a point on UV. U is a special case of the UV segment, so we'll just examine the latter. Let the farthest point be x away from U. Then, (1) by $STYZ$ and $STUV$ or $WXYZ$ and $UVWX$, $AB^2 = (7-x)^2 + 20^2$. (2) By $STZY$ and $TUXY$ and $STUV$, $AB^2 = 17^2 + (10+x)^2$. (3) By $STZY$ and $SVWZ$ and $STUV$, $AB^2 = 13^2 + (20-x)^2$. Setting AB from 1 and 2 equal gives 30/17. Setting (2) and (3) equal gives 3, which is clearly closer to A by route (1) than 30/17 is, so is clearly not the answer. Setting (1) and (3) equal gives 60/13, which is clearly closer than 30/17 by route 1. So, 30/17 is the answer.

25. Let $[a, b]$ and (a, b) denote the greatest common factor and least common multiple of a and b, respectively. From the fact that $[a, b](a, b) = ab$, we get $AB = 300 \times 135000$, $BC = 150 \times 81000$, and $AC = 6750 \times 202500$. The solution to this system is $A = 20250$, $B = 67500$, and $C = 600$. Thus, $A + B + C = 88350$.

26. Define "winning" as getting a string of at least four heads before a string of at least two tails and "losing" otherwise. Several winning sequences are HHHH... (with probability 1/16), THHHH... (1/32), HTHHHH... (1/64), XHTHHHH... (1/64), XXHTHHHH... (3/256), and so on. Some losing sequences are TT... (1/4), HTT... (1/8), XHTT... (1/8), XXHTT... (3/32), etc. Clearly, each of the former is 1/8 as likely as each of the latter. So, we can call the probability of winning in this way W, which means the probability of losing in this way is $8W$. Thus, the total probability of winning is $1/16 + 1/32 + W$, and for losing is $1/4 + 8W$. We also know that the game will be eventually won or lost (the probability of avoiding both forever is zero). Therefore, we have $1/16 + 1/32 + W = 1 - (1/4 + 8W) \to W = 7/96$. The overall probability of winning is $1/16 + 1/32 + W = 1/6$.

27. Without loss of generality, let O be the unit circle centered at the origin with A at $(1, 0)$. If F has polar coordinates of (r, θ), then the area of $\triangle OFA$ is given by $A(r, \theta) = r|\sin\theta|/2$. It follows that the desired probability is A divided by the area of the circle, π. To sum up the probabilities for all possible values for r and θ, we

integrate over the entire region:

$$\iint_R \frac{A(r,\theta)}{R}\,dA = \int_0^{2\pi}\int_0^1 \frac{r|\sin\theta|}{2\pi}\,r\,dr\,d\theta = \left(\int_0^{2\pi} |\sin\theta|\,d\theta\right)\left(\int_0^1 \frac{r^2}{2\pi}\,dr\right) = \frac{2}{3\pi}$$

Dividing this value by the area of the circle results in a probability of $2/(3\pi^2)$.

28. Let $p = \sqrt{3}$, $q = \sqrt{6}$, $r = 1$, $a = \sqrt{5}$, $b = \sqrt{2}$, and $c = \sqrt{5}$. By Euler's Tetrahedron Formula, if we let the capitalized version of each variable equal the square of its lowercase counterpart (i.e. $P = p^2$, $Q = q^2$, and so on), the volume V is given by

$$288V^2 = \begin{vmatrix} 0 & P & Q & R & 1 \\ P & 0 & C & B & 1 \\ Q & C & 0 & A & 1 \\ R & B & A & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 6 & 1 & 1 \\ 3 & 0 & 5 & 2 & 1 \\ 6 & 5 & 0 & 5 & 1 \\ 1 & 2 & 5 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

The value of the determinant is 72 therefore, the volume is $\sqrt{72/288} = 1/2$.

29. You need to know that the Earth is closest to the Sun in the Northern Hemisphere Winter (Southern Summer) and that a planet moves faster when it's closer to the sun (celestial mechanics), so therefore, the Southern Summer is shorter than the Northern Summer.

30. Setting up the proper equations and using the given information to eliminate variables, we ultimately end up with $4x + y = 26$, where x and y are the number of people who have solved only problems B and C, respectively. The only solution to this equation in positive integers where $x > 2y$ (otherwise, the number of people that have solved problems B and C but not A would be negative) is $(x, y) = (6, 2)$. So the answer is 6.

31. There are two types of rectangles that can be formed; ones that contain the inner unlined rectangle and ones that don't. There are $\binom{3}{1}\binom{4}{1}\binom{3}{1}\binom{3}{1} = 108$ rectangles of the first type and by inclusion-exclusion,

$$\binom{9}{2}\binom{3}{2}+\binom{9}{2}\binom{3}{2}-\binom{3}{2}\binom{3}{2}+\binom{9}{2}\binom{4}{2}-\binom{4}{2}\binom{3}{2}+\binom{9}{2}\binom{3}{2}-\binom{3}{2}\binom{3}{2}-\binom{3}{2}\binom{4}{2}$$

or 486 rectangles of the second type, for a total of $108 + 486 = 594$ rectangles.

32. In order for the distance to be rational, the two points must either belong to the same horizontal or vertical segment or they must form a right triangle with legs of 3 and 4 parallel to the coordinate axes. There are $9\binom{5}{2} + 5\binom{9}{2} = 270$ ways to choose the points in the first case and $2(6 \times 1 + 5 \times 2) = 32$ ways for the second (we find the number of intervals with horizontal length of 3 and vertical length of 4, and vice versa; we multiply by 2 since there are two possible slopes for the line segment created by the points). Therefore, the probability is $(270 + 32)/\binom{45}{2} = 151/495$.

33. Consider the lattice cube $C = \{(x,y,z)|0 \leq x,y,z \leq 3\}$. As usual, we assign each lattice point a value equal to the number of ways to get from that point to $(3,3,3)$. Three of the faces will resemble Pascal's Triangle.

1	1	1	1
1	2	3	4
1	3	6	10
1	4	20	20

Therefore, the other three faces "start" with

1	4	10	20
4	8	18	
10	18	36	
20			

But, the last few terms can be contributed to by the faces next to this one (they are valid approach paths for these edges). So, along one edge, we get $20 + 18 + 18 = 56$, $56 + 36 + 36 = 128$, and $128 + 128 + 128 = 384$, which is the desired count.

34. Out of the 12 possible moves, there are $\binom{12}{3}$ ways to choose which ones will go in the x-direction, $\binom{9}{3}$ ways in the y-direction, $\binom{6}{3}$ ways in the z-direction, and so on. The answer is $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 369600$.

35. If the five leftmost digits are a, b, c, d, and e, we seek the number of solutions to the equation $a+b+c+d+e \leq 8$ in nonnegative integers. By the usual "stars-and-bars" technique, the answer is $\sum_{i=0}^{8}\binom{i+5-1}{5-1} = \binom{8+6-1}{6-1} = 1287$.

36. There might be more than one answer, one of them being $160 \div 5 + 2 \times 34 = 100$.

37. An integer with six positive integral factors is of the form a^5 or b^2c, where a, b, and c are prime. There are only two integers less than 1000 that satisfy the former: 2^5 and 3^5. For the latter, we find all values of c less than $1000/b^2$ that aren't equal to b. When all is said and done, the desired total is $2+52+28+11+7+4+3+2+1 = 110$.

38. Clearly, $\binom{9}{3} = 84$ is an upper bound. There are 8 choices of three points (collinear) which don't produce circles, so now the count is down to 76. But any time four (or more, but that doesn't happen for these points) points lie on a circle, that circle was counted too many times. There are four circles that go through really small squares, so $76 - 3(4) = 64$. There is one circle that goes through a slightly larger, tilted square (61), and one that goes through the biggest square (58). There are four which go through 2×1 rectangles (46), and four which go through "arcs," which leaves a total of 34 circles.

39. Notice that if $x > 0$, the particle reaches the point $(x, 1-x)$ after $(2x-1)^2$ seconds. The smallest odd square greater than 2001 is $45^2 = 2025$. Therefore, after this many seconds, the particle will be at $(23, -22)$. Since $2025 - 2001 = 24$, the desired location is $(23-24, -22) = (-1, -22)$.

40. $12(a^2+b^2+c^2) + 26ab + 74bc + 51ac = (3a+2b+12c)(4a+6b+c)$

41. Since $-3(36) + 2(-6) + 14 < 6$ and $-3(16) + 2(-2) + 14 < 18$, Home and Town are on the same side of The River. If we reflect Home to the other side of The River, we obtain the point $(-12, 26, -10)$. Thus, the minimal distance is

$$\sqrt{(-12-16)^2 + (26-(-2))^2 + (-10-18)^2} = 28\sqrt{3}$$

42. The direction vectors for A and B are $\mathbf{n_1} = [3, 1, -2]$ and $\mathbf{n_2} = [8, 1, -4]$, respectively. Therefore, a unit vector perpendicular to both lines is $\mathbf{N} = (\mathbf{n_1} \times \mathbf{n_2})/||\mathbf{n_1} \times \mathbf{n_2}|| = [-2, -4, -5]/\sqrt{45}$. Now a vector formed by taking one point from each line is $\mathbf{u} = [18+55, -2+18, -9-12] = [49, 13, -21]$. It follows that the shortest distance is $|\mathbf{N} \cdot \mathbf{u}| = 3\sqrt{5}$.

43. From the exponential form of a complex number, we find that $\cos z = \cosh iz = (e^{iz} + e^{-iz})/2 = 4$. Letting $x = e^{iz}$ and rearranging, we get $x^2 - 4x + 1 = 0$. By the Quadratic Formula, $x = e^{iz} = 2 \pm \sqrt{3}$. Thus, $z = -i\ln(2 \pm \sqrt{3})$. Since the cosine function is periodic with period 2π, we can obtain another solution by adding a multiple of 2π onto z. Thus, the general solution is $z = -i\ln(2 \pm \sqrt{3}) + 2\pi n$, where n is any integer.

44. Let $P(p)$ stand for the probability of player p winning in n games. Notice that

$$\frac{P(B)}{P(A)} = \frac{\binom{n}{n-3}(3/5)^n(2/5)^{n-3}}{\binom{n}{n-3}(2/5)^n(3/5)^{n-3}} = \frac{27}{8}$$

But we also have $P(A) + P(B) = 1$. Solve to obtain $P(B) = 27/35$.

45. There are five cases to consider: no 2's, one 2, ..., four 2's. If there are no 2's, then there are 7 ways to pick the lowest card in a particular suit, producing $7 \times 4 = 28$ straight flushes. If there's one 2, there are $\binom{4}{3} = 4$ ways to pick three other cards for each possible lowest card and one way to pick three other cards for the Jack, which gives 29 rank possibilities, which we multiply by 4 (ways to choose which 2 we got) and by 4 (ways to pick the suit of our "real" cards), giving 464 total ways to get this hand. With two 2's, there are $\binom{4}{2} = 6$ ways for each lowest card, 3 ways for the Jack, and 1 way for the Queen, giving 46 ranking possibilities, times $\binom{4}{2} = 6$ choices for which 2's we got, and 4 ways to pick the suit of our "real" cards, giving 1104 total ways to get this hand. With three 2's, there are $\binom{4}{1} = 4$ ways for each lowest card, $\binom{3}{1} = 3$ for the Jack, $\binom{2}{1} = 2$ for the Queen, and one for the King, giving 34 ways to do the rank. There are $\binom{4}{3} = 4$ ways to pick which 2's we got, and 4 ways to pick the suit of

our "real" cards, giving 544 total ways to get this hand. For four 2's, we just need one other card, and there are 11 possibilities. There are 4 ways to pick the suit of that card, giving 44 total ways to get this hand. Adding gives $44 + 544 + 1104 + 464 + 28 = 2184$.

46. This is Tom's phone number, and he should have put a message on his voicemail telling Mu Alpha Theta people that the answer to this problem is $17.2 - 3\ln 5$.

47. The first lattice point closest to the y-axis is $(7, 41)$. From here, we can simply keep adding 7 to the abscissa and subtracting 11 from the ordinate to generate the other points. The complete list is (7, 41), (18, 34), (29, 27), (40, 20), (51, 13), and (62, 6). There are six points.

48. If the major and minor axes are A and B, respectively, the area of the ellipse is given by $\pi(A/2)(B/2) = 2000\pi \to AB = 8000$. Since 8000 has 28 positive integral factors and $A > B$, the answer is $28/2 = 14$.

49. Let $A = (3, 5)$, $B = (6, 17)$, and $C = (15, 5)$. Let l be the line $10x - 24y + 1 = 0$, which intersects AC at D and BC at E. Let B' be the reflection of B through l and m the line through B' and E, which intersects AC at F. To find the desired volume, we need only rotate three triangles about l: ABE, ADE, and EFC. Let their centroids be G, H, and I, respectively. We then have the following:

$$G = \left(\frac{977}{126}, \frac{1763}{189}\right) \qquad H = \left(\frac{3062}{315}, \frac{1007}{189}\right) \qquad I = \left(\frac{383323}{26334}, \frac{1007}{189}\right)$$

By the standard formula, if $d(p, l)$ is the shortest distance between p and line l, we have $d(G, l) = 218/39$, $d(H, l) = 89/78$, and $d(I, l) = 1953/2717$. Applying the determinant formula for the area of a triangle with given vertices, we obtain $[ABE] = 1388/21$, $[ADE] = 2759/630$, and $[EFC] = 3844/13167$. Using the Theorem of Pappus, $V = 2\pi r A$, we obtain a volume of

$$2\pi\Big(d(G,l)[ABE] + d(H,l)[ADE] + d(I,l)[EFC]\Big) = \frac{804206655391\pi}{1073242170}$$

50. Note that P is on the hyperbola. Given a hyperbola $x^2/a^2 - y^2/b^2 = 1$, the area of the parallelogram formed by the asymptotes and the parallels to the asymptotes passing through P is a constant and is equal to $ab/2$ (as long as P lies on the hyperbola). The answer is $(4)(5)/2 = 10$.

51. Consider the power series $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$. Differentiating and multiplying both sides by x gives $\sum_{n=0}^{\infty} nx^n = x(1-x)^{-2}$. Proceeding this way—repeated differentiation and multiplication by x—until an n^3 is obtained, we get

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{x(2x+1)(1-x) + 3x(x^2+x)}{(1-x)^4}$$

Let $x = 1/2$ to get $\sum_{n=0}^{\infty} n^3/2^n = \sum_{n=1}^{\infty} n^3/2^n = 26$.

52. Factual questions; you either know them or not.

A. 1995 Convention in Maine

B. 1994

C. Trinity University, San Antonio Texas

53. The limit of a product is just the product of the individual limits so the answer is $THESKY$ or in words, "The sky's the limit." Double points for the latter.

54. $\displaystyle\sum_{i=1}^{n} i^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$

55. From an aerial view of the hallways, let θ be the angle the ladder makes with the 8-foot wide hallway at the corner. Using trigonometry, we find that the length of the ladder as a function of θ is given by $L(\theta) = 8\csc\theta + 4\sec\theta$. Taking $L'(\theta) = 0$ produces a critical value of $\theta = \arctan\sqrt[3]{2}$, a global maximum by the First Derivative Test. Thus, the length of the longest ladder is $L(\arctan\sqrt[3]{2}) = 4(1+2^{2/3})^{3/2} \approx 16.65$.

56. Let a, b, and c stand for the usual triangle sides and let $PA = x$, $PB = y$, and $PC = z$. Furthermore, let $\angle PAB = \angle PCA = \angle PBC = \omega$. From the Law of Cosines, $x^2 = b^2 + z^2 - 2bz\cos\omega$, $y^2 = c^2 + x^2 - 2cx\cos\omega$, and $z^2 = a^2 + y^2 - 2ay\cos\omega$. Adding all these equations and simplifying, we get $a^2+b^2+c^2 = 2\cos\omega(ay+cx+bz)$. Drawing the triangle, we notice that $ay\sin\omega + xc\sin\omega + bz\sin\omega = 2K$, where K is the area of $\triangle ABC$. Substituting, we get $a^2+b^2+c^2 = 2\cos\omega(2K/\sin\omega)$, or $a^2+b^2+c^2 = 4K\cot\omega$. Therefore, $\tan\omega = (4K)/(a^2+b^2+c^2) = (4\times 60)/(8^2+15^2+17^2) = 120/289$.

57. Denote the sum by S. Notice that

$$\frac{S}{6} = \frac{1}{36} + \frac{1}{216} + \frac{2}{1296} + \cdots$$
$$\frac{S}{36} = \frac{1}{216} + \frac{1}{1296} + \frac{2}{46656} + \cdots$$
$$\frac{S}{6} + \frac{S}{36} = \frac{1}{36} + \frac{2}{216} + \frac{3}{1296} + \cdots = S - \frac{1}{6}$$

Hence, $S/6 + S/36 + 1/6 = S$. Solve to obtain $S = 6/29$.

58. Given equilateral $\triangle ABC$, fold vertex A so that it hits BC at F and the crease makes corners D and E along AB and AC, respectively. Let $FD = x$ and $FE = y$. From the Law of Cosines, $x^2 = (5-x)^2 + 4^2 - 2(5-x)(4)\cos 60^\circ \rightarrow x = 7/2$ and $y^2 = (5-y)^2 + 1^2 - 2(5-y)(1)\cos 60^\circ \rightarrow y = 7/3$. Applying the Law of Cosines to $\triangle DEF$, we find that the length of the crease is equal to $\sqrt{x^2+y^2-2xy\cos 60^\circ} = 7\sqrt{7}/6$.

59. Rewrite the equation as $(2x-4)^4 + (9-3x)^4 = (5-x)^4$ and put $a = 2x-4$ and $b = 9-3x$. The equation then becomes $a^4+b^4 = (a+b)^4$, or $ab(2a^2+3ab+2b^2) = 0$, which implies that x equals 2 or 3. The factor $2a^2+3ab+2b^2$ is easily shown to contain no real roots so the desired product is $2\times 3 = 6$.

60. Place the first point A on the circle; it doesn't matter. Now consider the three other points B, C, and D. There are $\binom{3}{3}(2) = 2$ ways for them to end up on the same side of the first point and $\binom{3}{2}(2) = 6$ ways for the points to be split in a 2:1 ratio, i.e. two points on one side, one point on another. If all the points are on the same side, the probability is $(1/6)^3 = 1/216$. Suppose that they're split 2:1 and B is the lone point. Let x be the position of B; clearly $0 < x < 1/6$. Now let y be the position of either C or D; its range is $0 < y < 1/6 - x$. Thus, the probability is

$$2\left(\frac{1}{216}\right) + 6\left(\int_0^{1/6}\left(\frac{1}{6} - x\right)^2 dx\right) = \frac{1}{54}$$

61. This is a binary code, similar to those discussed in *Contact*, by Carl Sagan. The total number of digits is 1333, which is the product of exactly two primes, 31 and 43. This would perhaps lead one to arrange them in a 31 by 43 grid, the entire border of which is 1s. If we transfer the code to graph paper, filling in a square for 1 and leaving the square blank for 0, one would see a picture that looked like $2e^{\pi i/3}$. Any form of this number is acceptable.

62. When decrypted, the numbers turn into the following message:

> *This message was inspired by the Beale Cipher, a century-old code which supposedly contains directions to a buried treasure estimated to be worth twenty million dollars. Hopefully the references to Thomas Beale and various things in the Goblet of Fire throughout this test were enough to tip you off as to what kind of encryption was used and which document acted as the key for this one. Okay, on to the problem: one-and-a-half chickens lay one-and-a-half eggs in one-and-a-half days; how many days will it take one chicken to lay one egg?"*

The answer is, of course, $(3/2)(1)/(1) = 3/2$.

63. The puzzle, if done correctly, should resolve into a crude drawing of

$$\int_1^4 \frac{dx}{117}$$

giving an answer of $(4 - 1)/117 = 1/39$.