Mu Alpha Theta National Convention: Denver 2001
Alpha Individual Test – Solutions

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1. **(B)**. $g(f(x)) = 2(x^2 + 3x + 4) + 3 = 2x^2 + 6x + 11$.

2. **(A)**. Brian has gained a $5(1/2) = 5/2$ mile head start so it will take Keith $(5/2)/(7-5) = 5/4$ hours to catch up, or 75 minutes.

3. **(B)**. $\sqrt[5]{a^{12}b^{34}c^{56}} = a^2b^6c^{11}\sqrt[5]{a^2b^4c}$

4. **(A)**. $(3-4i)(4+i) = 12 + 3i - 16i - 4i^2 = 16 - 13i$

5. **(C)**. We have $x = 10^k + 1$ so x simply consists of two 1s with $k-1$ zeroes in-between. Thus, $y = x$ so their ratio is $1:1$.

6. **(B)**. Let (a, b, c) be the sides of the triangle, with $a \le b < c$. From the given information, we have $b = 2 + 2a$ and $c = b + 1 = 2a + 3$. By the Pythagorean Theorem, $a^2 + (2+2a)^2 = (2a+3)^2$. Solving for the positive value of a, we get $a = 5$ so $b = 12$, and $c = 13$. The perimeter of the triangle is $5 + 12 + 13 = 30$.

7. **(C)**. The sum and product of the roots are $-(-5)/1 = 5$ and $9/1 = 9$, respectively. So we have $(a-2)(b-2) = ab - 2(a+b) + 4 = 9 - 2(5) + 4 = 3$.

8. **(C)**. The increment in the x-direction is $(5/6 - 1/2)/3 = 1/9$. The x-coordinates of the points of trisection is then $1/2 + 1/9 = 11/18$ and $11/18 + 1/9 = 13/18$. Their sum is $11/18 + 13/18 = 4/3$.

9. **(C)**. By the proper sum formula, $\sum_{n=1}^{7}(5n+2) = (7/2)(7+37) = 154$.

10. **(B)**. Note that $5063\pi/6 = 843\pi + 5\pi/6$, which is coterminal to $\pi + 5\pi/6 = 11\pi/6$.

11. **(C)**. Let p and f denote the number of people who've passed and failed the final, respectively. Thus, we have the system $p + f = 40$ and $81p + 66f = (75)(40)$. Eliminate f by substituting $f = 40 - p$ into the second equation to get $p = (9)(40)/15 = 24$.

12. **(B)**. Take the square root of both sides of the equation and get $\sin x = \pm 1$. The values of x that work in the given interval are $\pi/2$ and $3\pi/2$.

13. **(C)**. There are $\binom{14}{2}$ ways to assign people to Gemini. Once that's taken care of, there are $\binom{12}{2}$ possible two-student teams left to take Proofs. The total number of possible assignments is $\binom{14}{2}\binom{12}{2} = 6006$.

14. **(D)**. Write the equation as $\tan x + \sin x - \sec x - 1 = 0$ or $(\sin x - 1)(\sec x + 1) = 0$. If $\sin x = 1$, $x = \pi/2$, which causes the original equation to be undefined. If $\sec x = -1$ then $x = \pi$. There is only one solution.

15. **(D)**. By the Law of Cosines, $AC = \sqrt{3^2+3^2-2(3)(3)\cos 120°} = 3\sqrt{3}$. The region common to $\triangle ACE$ and $\triangle BDF$ is a regular hexagon with side length $AC/3 = \sqrt{3}$. Since all regular hexagons are similar to each other, the desired ratio is the square of the ratio of the sides, or $(\sqrt{3}/3)^2 = 1/3$.

16. **(A)**. The largest angle will be opposite the longest side; call this angle θ. By the Law of Cosines, $\cos\theta = (3^2+7^2-8^2)/(2(3)(7)) = -1/7$. Thus, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-1/49} = 4\sqrt{3}/7$. Note that we choose the positive square root for the sine of any angle in a triangle is positive.

17. **(D)**. $\left(\dfrac{x+y}{x-y}\right)^2 = \dfrac{x^2+y^2+2xy}{x^2+y^2-2xy} = \dfrac{8xy}{4xy} = 2 \rightarrow \dfrac{x+y}{x-y} = \sqrt{2}$

18. **(E)**. Simplifying the left side of the equation leads to

$$\left(\frac{\sin^2 x}{\tan^4 x}\right)^3\left(\frac{\csc^3 x}{\cot^6 x}\right)^2 = \left(\frac{\sin^6 x}{\tan^{12} x}\right)\left(\frac{\tan^{12} x}{\sin^6 x}\right) = 1$$

giving us the true statement $1 = 1$. Thus, there are infinitely many solutions.

19. **(D)**. Complete the square for each equation:

I. $3(x-1)^2 + 2(y+3)^2 = 0$
II. $2(x+7)^2 - (y-1)^2 = 2$
III. $9(x-5)^2 - 4(y-6)^2 = 0$
IV. $2(x+2)^2 + 2(y-6)^2 = 0$

All are degenerate conics, except for II (it's a hyperbola).

20. **(D)**. By the standard formula, $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = (3)(4)\cos 225° = -6\sqrt{2}$.

21. **(B)**. Partition the set into 107 compartments of three consecutive integers, as follows

$$\{1,2,3\},\{4,5,6\},\{7,8,9\},\ldots,\{319,320,321\}$$

Notice that if we pick 215 distinct numbers from this set, at least one of the compartments will be empty, i.e. we'll have a set of three consecutive numbers. Therefore, the probability is 1.

22. **(D)**. By the Binomial Theorem, the expression is equal to $(\cos^2 x + \sin^2 x)^3 = 1^3 = 1$.

23. **(C)**. The diameter of the circle is equal to the length of the altitude to the hypotenuse; call this value h. Thus, $h = (BC)(AC)/AB = (3)(4)/5 = 12/5$. Therefore, the radius is $h/2 = 6/5$ and the area of the circle is $\pi(6/5)^2 = 36\pi/25$.

24. **(A)**. Drawing the polar curves, we see that there are three intersection points: $(1, \pi/2)$, $(1, 3\pi/2)$, and $(1, \pi)$. Notice that setting the r-values equal to each other will produce the first two points, but not the third. This emphasizes the importance of drawing a graph to find intersection points.

25. **(C)**. Break up f as $f(a,b,c) = 3\log_x c + 2\log_x b + \log_x a$. Since $x > 1$, if $u > v$ then $\log_x u > \log_x v$ which means $\alpha > \gamma > \beta$. Thus, let $c = 7$, $b = 5$, and $a = 3$ for a maximum value of $3\alpha + 2\gamma + \beta$.

26. **(A)**. Group the sum as $\left(\arctan\frac{8}{13} + \arctan\frac{5}{21}\right) + \left(\arctan\frac{1}{4} + \arctan\frac{3}{5}\right)$. Apply the formula $\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$ to each pair to get $\arctan 1 + \arctan 1 = \pi/2$.

27. **(A)**. Standard Deviation $= (4-3)\sqrt{\left(\dfrac{36}{36+64}\right)\left(\dfrac{64}{36+64}\right)} = \dfrac{12}{25}$

28. **(D)**. $\sin\dfrac{\pi}{6} + \cos\dfrac{2\pi}{3} - \tan\dfrac{3\pi}{4} = \dfrac{1}{2} - \dfrac{1}{2} + 1 = 1$

29. **(A)**. Treat the problem like "regular" multiplication, except carry out 9s, not 10s:

$$\begin{array}{r} 865 \\ \times\ 71 \\ \hline 865 \\ 67080 \\ \hline 68055 \end{array}$$

30. **(D)**. Use a double-angle formula and rewrite the equation in terms of the sine: $\sin 3x = \cos 6x = 1 - 2\sin^2 3x$. Factor to get $(\sin 3x + 1)(2\sin 3x - 1) = 0$. Thus, $\sin 3x = -1$ or $\sin 3x = 1/2$. On the specified interval, the first equation has solutions $\{\pi/2, 7\pi/6, 11\pi/6\}$ while the second has roots of $\{\pi/18, 13\pi/18, 25\pi/18, 5\pi/18, 17\pi/18, 29\pi/18\}$. Their total sum is $17\pi/2$.

31. **(A)**. We use the properties of hyperbolic functions to simplify the math. The expression is equal to $\sqrt{1+\sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x = (e^x + e^{-x})/2$.

32. **(C)**. Since θ is in Quadrant II, $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-5/9} = -2/3$. By the triple-angle formula, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4(-2/3)^3 - 3(-2/3) = 22/27$.

33. **(B)**. The order of the first matrix is 2×3 while the second 3×1 so the order of the resulting product is 2×1. Performing the usual row-by-column dot product, we get

$$\begin{pmatrix} 7 & -1 & 3 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 14-4+18 \\ 4+0+30 \end{pmatrix} = \begin{pmatrix} 28 \\ 34 \end{pmatrix}$$

The sum of the elements is 62.

34. **(B)**. The sum of all multiples of 7 less than 114 is $7+14+21+\cdots+112 = 7(16)(17)/2 = 952$. The sum of the first 114 natural numbers is $(114)(115)/2 = 6555$. The answer is $6555 - 952 = 5603$.

35. **(C)**. If n is an integer, notice that $n(n+1)(2n+1)/6$ and $n^2(n+1)^2/4$ are also integers since they represent the sum of the sum of the first n squares and cubes, respectively. Thus, their product, $n^3(n+1)^3(2n+1)/24$, is also an integer. Consequently, $n^3(n+1)^3(2n+1) = (n^2+n)^3(2n+1)$ is always divisible by 24.

36. **(D)**. Isolating the cosine function and squaring both sides, we get $\cos^2 x = 1 - \sin^2 x = (6/5 - \sin x)^2$, which, after some algebra, simplifies to $50u^2 - 60u + 11 = 0$, where $u = \sin x$. By the Quadratic Formula, $u = (6 \pm \sqrt{14})/10$. Since $0 \le x \le \pi/6$, $0 \le \sin x \le 1/2$. Thus, minus wins over plus and $u = \sin x = (6 - \sqrt{14})/10$.

37. **(B)**. The direction vectors for the lines are $[-2,1,1]$ and $[-3,4,-1]$. The lines are contained in the plane which means the cross product of the direction vectors gives the vector perpendicular to the plane. Evaluating, we get $[-2,1,1] \times [-3,4,-1] = [-5,-5,-5]$ or after discarding scalar multiples, $[1,1,1]$. Since $(1,4,0)$ is a point on the plane, its equation is $(x-1)+(y-4)+z=0$ or $x+y+z=5$.

38. **(C)**. Expand the left side to get $(a+b+c)(a+b-c) = (a+b)^2 - c^2 = a^2+2ab+b^2-c^2 = 3ab \to (a^2+b^2-c^2)/(2ab) = 1/2$. From the Law of Cosines, $\cos C = (a^2+b^2-c^2)/(2ab)$ so we have $\cos C = 1/2$ or $\sec C = 2$.

39. **(A)**. We only have to consider the first four whole numbers as afterwards, the remainders will repeat with a period of 4: $0^3 \equiv 0 \pmod 4$, $1^3 \equiv 1 \pmod 4$, $2^3 \equiv 0 \pmod 4$, and $3^3 \equiv 3 \pmod 4$. In one period, two of the remainders will be congruent to either 1 or 3 $\pmod 4$. Since S has $1136/4 = 284$ full periods, there are $2(284) = 568$ members in S with the desired property.

40. **(B)**. The coefficient of x^2 and y^2 aren't the same but are equal in sign. Thus, the conic is an ellipse.

41. **(C)**. Combine the left-hand logarithm and multiply both sides of the equation by 4 to get $4\ln(x/y) = 4\log_y x \to \ln(x^4/y^4) = \log_y x^4 \to \ln y^5 = \log_y y^9 = 9$. Thus, $y = e^{9/5}$ and $x = y^{9/4} = e^{81/20}$.

42. **(D)**. If the numbers are x and y, $x+y=60$ and $xy=62$. Thus, $x^2+y^2 = (x+y)^2 - 2xy = (60)^2 - 2(62) = 3476$.

43. **(A)**. The situation at hand can be modelled by the system

$$\begin{aligned} n + d + q &= 162 \\ 5n + 10d + 25q &= 2200 \\ n &= q - 12 \end{aligned}$$

Solve to get $(n,d,q) = (40,70,52)$; George has 70 dimes.

44. **(A)**. If $\cos x = 1/3$, then $\sin x = \sqrt{1 - 1/9} = 2\sqrt{2}/3$, for $\cos x < \sin x$. Substitute the repeating portion into the equation to get $u = 2\sin x + (\cos^2 u)/u$ or $u^2 - 2u\sin x - \cos^2 x = 0$. As a quadratic in u, the solution is $u = \sin x \pm 1 = 2\sqrt{2}/3 \pm 1$. Since u is positive (why?), the answer is $2\sqrt{2}/3 + 1$.

45. **(E)**. Bring everything to one side, simplify, and factor to get $r(r-9)/(r^2-9) < 0$. Test numbers around the critical values of $r = -3, 0, 3, 9$ to get a solution set of $-3 < r < 0$ or $3 < r < 9$.

46. **(A)**. Let the numbers be x and $486 - x$ where $x < 486 - x$. The problem states that $486 - x = x + 100$. Solve to get $x = 193$.

47. **(D)**. Call the dimensions l, w, and h; we want the value of lwh. From the given information, $lw = 64$, $lh = 80$, and $wh = 20$. Multiply all these to obtain $(lw)(lh)(wh) = l^2w^2h^2 = (64)(80)(20)$. Take the square root of both sides to get $lwh = \sqrt{(64)(80)(20)} = 320$.

48. **(C)**. The equation $\log N = \lfloor \log N \rfloor$ asserts that $\log N$ is equal to its integer part, which means that $\log N$ must be itself an integer. This will only happen if $N = 10^k = 2^k 5^k$, where k is a nonnegative integer. The number of positive integers relatively prime to N is given by the totient function $\phi(x)$. By the formula for $\phi(x)$, we have $\phi(N) = \phi(2^k 5^k) = 2^k 5^k (1 - 1/2)(1 - 1/5) = 2^k 5^k (2/5) = 2N/5$. Thus, the desired percentage is $(2N/5)/(N) \times 100\% = 40\%$.

49. **(D)**. $17 + 9d = 167 \rightarrow d = 50/3$

50. **(D)**. By Heron's Formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$. Since $s = (a+b+c)/2$, we can explicitly write A in terms of its sides. Some algebra produces

$$A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{b+c-a}{2}\right)}$$

or $4A = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$. Thus, the quantity we wish to evaluate is simply 4 times the area of a triangle with side lengths $\sqrt{2}$, $\sqrt{13}$, and $\sqrt{17}$. Now consider the points $(0,0)$, $(1,4)$, and $(2,3)$. Notice that the triangle with these vertices has exactly the lengths we want. By the determinant formula

$$A = \pm\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \frac{5}{2}$$

which means $4A = 4(5/2) = 10$.