Alpha Gemini Nationals 2015
Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. Two sides of a triangle have lengths 4 and 5. If the area of the triangle is 6 what is the smallest possible length of the 3rd side?

(A) $\dfrac{5}{2}$ (B) $\dfrac{3}{2}$ (C) 3   (D) 2   (E) NOTA

2. Find the value of $f\left(\dfrac{-2}{\sqrt{3}}\right)$ if $f(x) = Arc\sec(x)$.

(A) $\dfrac{5\rho}{6}$ (B) $\dfrac{2\rho}{3}$ (C) $-\dfrac{\rho}{6}$ (D) $-\dfrac{\rho}{3}$ (E) NOTA

3. What is the units digit of $\displaystyle\sum_{n=2}^{8} n^n$?

(A) 0   (B) 4   (C) 6   (D) 8   (E) NOTA

4. If $f(x) = \dfrac{x}{1+2x}$ and $g(x) = f(f(x))$, then what is the inverse function of $y = g(x)$?

(A) $\dfrac{x}{4x+1}$ (B) $\dfrac{-x}{4x+1}$ (C) $\dfrac{x}{4x-1}$ (D) $\dfrac{-x}{4x-1}$ (E) NOTA

5. What is the coefficient of the x in the expansion of $\left(\dfrac{x}{2} - \dfrac{4}{\sqrt{x}}\right)^7$?

(A) 1120   (B) -5120   (C) $-\dfrac{1}{32}$ (D) -140   (E) NOTA

6. What are the sum of the distinct roots of $0 = \sin x + \sin 2x + \sin 3x$ on the interval $0 < x < 2\rho$?

(A) ρ (B) 3ρ (C) 5ρ (D) 7ρ (E) NOTA

7. If the cross product of the two vectors $a = \langle 4, -11, 10 \rangle$ and $b = \langle x, -2x, -3x \rangle$ satisfies the equation $a \times b = \langle y, 154, z \rangle$, what is the dot product of a and b?

(A) 0   (B) 217   (C) -63   (D) $\dfrac{119}{2}$ (E) NOTA

8. Evaluate the following limit: $\lim\limits_{x \to 0^+} \dfrac{1 - \dfrac{1}{5x}}{2 - \dfrac{2}{3x}}$

(A) 0   (B) $\dfrac{3}{10}$ (C) $\dfrac{1}{2}$ (D) ¥   (E) NOTA

9. What is the range of the following function: $y = \left|\ln\left(5 - 2x^2\right)\right|$?

(A) $0 < y \pounds \ln 5$ (B) $0 \pounds y \pounds \ln 5$ (C) $-\sqrt{2} \pounds y \pounds \sqrt{2}$ (D) $-\sqrt{2} < y < \sqrt{2}$ (E) NOTA

10. What value of the k could make the roots of the quadratic equation below equal?
$$2x^2 - 60 - k(4x - 22) = 0$$

(A) -3   (B) -4   (C) -5   (D) 6   (E) NOTA

11. If $_nP_r = 1680$ and $_nC_r = 70$, then what is the value of $n - r$?

(A) 2   (B) 3   (C) 4   (D) 5   (E) NOTA

12. What is the value of $\sin\left(Arc\sec\left(-\dfrac{41}{9}\right)\right)$?

(A) $\dfrac{9}{40}$ (B) $-\dfrac{40}{41}$ (C) $-\dfrac{9}{40}$ (D) $\dfrac{9}{41}$ (E) NOTA

13. How many positive three digit odd integers do not contain the digits 1 or 4?

(A) 360   (B) 320   (C) 280   (D) 224   (E) NOTA

14. Convert the Cartesian coordinates $(-6\sqrt{3}, 6)$ to polar coordinates.

(A) $\left(12, \dfrac{5\pi}{6}\right)$ (B) $\left(12, \dfrac{11\pi}{6}\right)$ (C) $\left(6, \dfrac{\pi}{6}\right)$ (D) $\left(6, -\dfrac{\pi}{6}\right)$ (E) NOTA

15. If a complex number $z = \dfrac{\dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{3}i} - \dfrac{2\sqrt{3}i}{3\sqrt{2} + 2\sqrt{3}i}$, what is the value of $|z|$?

(A) $\dfrac{\sqrt{6}}{2}$ (B) $2\sqrt{6}$ (C) $\sqrt{2}$ (D) 1   (E) NOTA

16. A sphere is inscribed in a right cicular cone that has a slant height of 5. The radius of the cone is 3, what is the radius of the inscribed sphere?

    (A) 2   (B) 3   (C) 4   (D) $\sqrt{2}$ (E) NOTA

17. What is the axis of symmetry of the parabola defined by the parametric equations $x = \dfrac{t+4}{3}$ and $y = \dfrac{t^2}{2} - t$?

    (A) $x = \dfrac{5}{6}$ (B) $x = \dfrac{4}{3}$ (C) $x = \dfrac{5}{3}$ (D) $x = -1$ (E) NOTA

18. The perimeter of a circular sector is equal to one-third the length of the circumference of the circumference of the circle from which this sector was taken. If the area of the sector is $16\left(\dfrac{\pi}{3} - 1\right)$, then what is the radius of the sector?

    (A) $3\sqrt{3}$ (B) 4   (C) 16   (D) 27   (E) NOTA

19. What is the first quadrant area enclosed by an ellipse whose equation is $y = \pm\dfrac{3}{2}\sqrt{16 - x^2}$?

    (A) $\dfrac{3}{2}\pi$ (B) 6π (C) 12π (D) 24π (E) NOTA

20. If $(3x + 1)$ leaves no remainder when divided into $3x^3 - 5x^2 + Ax + 2$, then what is A?

    (A) $-\dfrac{14}{3}$ (B) 4   (C) 0   (D) -6   (E) NOTA

21. What is the trace of the inverse of $\begin{bmatrix} 3 & 9 & 9 \\ 3 & 12 & 9 \\ 3 & 9 & 12 \end{bmatrix}$.

    (A) 0   (B) 1   (C) 2   (D) 3   (E) NOTA

22. What is the value of $\left((i^{-3} + i^3)^{-3}\right)^3$?

    (A) $-\dfrac{1}{8}i$ (B) $\dfrac{1}{64}i$ (C) $-\dfrac{1}{64}i$ (D) $-\dfrac{1}{512}i$ (E) NOTA

23. If the equation $(a + 6)x^2 - 8x + a = 0$ has two distinct real roots, how many integer values of a satisfy the equation?

    (A) eight   (B) six   (C) four   (D) two   (E) NOTA

24. Let F be the Cartesian form of a line of the following vector equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$ What is F?

    (A) $4x - y = 0$ (B) $3x - 4y = 0$ (C) $y = -4$ (D) $4x + 3y = 16$ (E) NOTA

25. Using F from the previous problem, #24, let G be the equation of the normal line to F which also goes through the x-intercept of F. What is the y-intercept of G?

    (A) $\dfrac{3}{4}$ (B) $-\dfrac{3}{4}$ (C) -3   (D) DNE   (E) NOTA

26. Solve for x: $\dfrac{\log(0.04)}{\log(0.25)} = \dfrac{\log(5^x)}{\log(4)}$

    (A) 1   (B) -1   (C) 2   (D) $\log_2 5$ (E) NOTA

27. Evaluate $\displaystyle\sum_{x=0}^{\infty} \dfrac{4}{5^{x-1}}$.

    (A) $\dfrac{5}{4}$ (B) 25   (C) 100   (D) 20   (E) NOTA

28. Find the least common multiple of 110_3 and 121_4.

    (A) 606_7 (B) 453_8 (C) 361_9 (D) 297   (E) NOTA

29. Simplify completely: $\dfrac{\sec x \csc x - \tan x \csc x}{(\sec x - \tan x)^2 + 1}$

    (A) $2\tan x$ (B) $2\cos x$ (C) $\dfrac{1}{2}\cot x$ (D) $\dfrac{1}{2}\cos x$ (E) NOTA

30. What is the x-coordinate of the minimum of the parabola given by the equation

$$\dfrac{y+3}{\left(-\dfrac{2}{5}x + 1\right)} = 5x?$$

    (A) $\dfrac{1}{8}$ (B) $\dfrac{5}{4}$ (C) $\dfrac{5}{2}$ (D) -3   (E) NOTA