2022 Mu Alpha Theta National Convention **Washington DC**

Mu Limits and Derivatives Solutions

1.	B	Using quotient and product rule: $\frac{(ab)'cd-ab(cd)'}{c^2d^2} = \frac{a'bcd+ab'cd-abc'd-abcd'}{c^2d^2}$
2.	B	Slope of the line is $-\frac{1}{2022}$. Therefore, the inverse reciprocal is 2022.
3.	E	$g'(x) = \frac{-\left(f^{-1}(x)\right)'}{\left(f^{-1}(x)\right)^2} = \frac{-1}{f'\left(f^{-1}(x)\right)\left(f^{-1}(x)\right)^2}, g'(2) = -\frac{1}{(1)^2 f'(1)} = \frac{1}{4}$
4.	C	$\ln(2022) - \ln(x^{2022}) - \ln\left(tan^{-1}\left(\frac{1}{x^{2022}}\right)\right) = \ln(2022) - \ln\left(x^{2022}\, tan^{-1}\left(\frac{1}{x^{2022}}\right)\right) =$ $\ln(2022) - \ln\left(\frac{tan^{-1}\left(\frac{1}{x^{2022}}\right)}{x^{-2022}}\right)$. Applying L'Hosptial's rule, $\lim_{x\to\infty} \ln(2022) - \ln\left(\frac{tan^{-1}\left(\frac{1}{x^{2022}}\right)}{x^{-2022}}\right) =$ $\lim_{x\to\infty} \ln(2022) - \ln\left(\frac{-2022x^{-2023}}{-2022x^{-2023}(1+(x^{-2022})^2)}\right) = \ln 2022$
5.	A	At $(0,2022), t = 1, \frac{dx}{dt} = \frac{1}{t}\|_{t=1} = 1, \frac{dy}{dt} = 2t\|_{t=1} = 2, \frac{dy}{dx} = 2$. Therefore, the tangent line is $y - 2022 = 2x \to y = 2x + 2022$
6.	D	$\ln f(x) = \ln 60 + \frac{1}{5}\ln(-x^2+2) + \ln\tan^{-1} x - \frac{1}{3}\ln x - \frac{1}{2}\ln(2x^2-1)$ $\frac{f'(x)}{f(x)} = -\frac{2x}{5(-x^2+2)} + \frac{1}{\tan^{-1} x\,(1+x^2)} - \frac{1}{3x} - \frac{2x}{2x^2-1}, f(-1) = 15\pi$ $f'(-1) = 15\pi\left(\frac{2}{5} - \frac{2}{\pi} + \frac{1}{3} + 2\right) = 41 - 30\pi$
7.	C	Using Taylor series, $$\lim_{x\to0}\frac{\cos x^2 - e^{x^4}}{\sin x^4} = \lim_{x\to0}\frac{1 - \frac{x^4}{2} + \frac{x^8}{4!} + \cdots - 1 - x^4 - \frac{x^8}{2!} - \cdots}{x^4 - \frac{x^{12}}{3!} + \cdots}$$ $$= \lim_{x\to0}\frac{-\frac{1}{2} + \frac{x^4}{4!} + \cdots - 1 - \frac{x^2}{3!} - \cdots}{1 - \frac{x^8}{3!} + \cdots} = -\frac{3}{2}$$
8.	A	We can see that $\lim_{x\to0} f(x) = \infty$, so we need a x^n such that $\lim_{x\to0}\frac{x^n}{e^{x^{2022}}-1} = L$. Assuming an indeterminate form, applying L'Hospital to get rid of the indeterminate form, we have $\lim_{x\to0}\frac{nx^{n-1}}{2022x^{2021}e^{x^{2022}}} = \frac{nx^{n-1}}{2022x^{2021}}$. Thus, to cancel the x^{2021} term, we can set $n = 2022$ so $x^{-n} = x^{-2022}$
9.	E	$\frac{1}{R} = \left\|\frac{f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}}\right\| = \frac{\cosh x}{(1+\sinh^2 x)^{\frac{3}{2}}} = \frac{\cosh x}{\cosh^3 x} = \left(\frac{2}{e^x+e^{-x}}\right)^2\|_{x=\ln 2} = \frac{16}{25}. \to R = \frac{25}{16}$
10.	B	$\frac{d}{dx}\left[2022!^{2022!} + \ln(2022!^x) + \ln 2022 - e^{2022} - 2022^x\right]\Big\|_{x=0} = \frac{d}{dx}\left[x\ln(2022!) - 2022^x\right]\Big\|_{x=0} = \ln(2022!) - \ln(2022)\, 2022^0 = \ln(2021!)$
11.	B	$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{8i}{n^2} + \frac{2}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\frac{2i}{n} + 1\right)^2 \to \Delta x = \frac{2}{n} = \frac{b-a}{n}, a = 1 \text{ so } b = 3, f(x) = x^2$ $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{8i}{n^2} + \frac{2}{n}\right) = \int_1^3 x^2\, dx = \frac{26}{3}$.

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12.	D	Using l'Hospital, we get $\lim_{x\to 0}\frac{g(x^2+3x)}{\sin x} = \lim_{x\to 0}\frac{g'(x^2+3x)(2x+3)}{\cos x} = \frac{4(3)}{1} = 12$.
13.	B	To ensure continuity, we need $\lim_{x\to 0}\frac{9x-3\sin(3x)}{5x^3} = \lim_{x\to 0}\left(ke^x - x^{\frac{1}{3}} + 1\right)$. Using l'Hospital three times on the first limit, we get $\lim_{x\to 0}\frac{9x-3\sin(3x)}{5x^3} = \lim_{x\to 0}\frac{9-9\cos(3x)}{15x^2} = \lim_{x\to 0}\frac{27\sin(3x)}{30x} = \lim_{x\to 0}\frac{81\cos(3x)}{30} = \frac{81}{30} = \frac{27}{10}$. The second limit is easily calculated to be $k+1$. Hence, $k = \frac{17}{10}$ and so $a + b = 17 + 10 = 27$.
14.	C	$\left(\frac{1}{\sqrt{1+x^{2022}}}\right)' = -\frac{1}{2}\cdot\frac{\left(1+x^{2022}\right)'}{(1+x^{2022})^{\frac{3}{2}}} = -\frac{1011x^{2021}}{(1+x^{2022})^{\frac{3}{2}}}$
15.	C	$x\ln(2022x) + \frac{1}{2022e^{2022x}} = x\ln(2022) + x\ln x + \frac{e^{-2022x}}{2022}$. Therefore, the derivative is calculated as: $\ln(2022) + \ln x + 1 - e^{-2022x} = \ln(2022x) + 1 - e^{-2022x}$
16.	A	$f(x) = x^2 - 2022 \to f(45) = 3, f'(x) = 2x \to f'(45) = 90, x - \frac{f(x)}{f'(x)}\|_{x=45} = 45 - \frac{1}{30} = 44.9666\ldots$
17.	C	The coordinates that represent the biker can be modeled as $(100\cos\theta, 100\sin\theta)$ and $(100\cos\theta, -100\sin\theta)$, where $\theta = \frac{s}{r}, s \to$ arc length, $r \to$ radius so $\theta' = \frac{s'}{r} = \frac{5}{100} = \frac{1}{20}$ Therefore, the distance $d = 200\sin\theta \to d' = 200\,\theta'\cos\theta = 10\cos\frac{5\cdot 5}{100} = 10\cos\frac{1}{4}$
18.	A	Using implicit differentiation, we get $18(1+y')(x+y) + 2(1-y')(x-y) = 0, (x,y) = (1,1) \to y' = -1$. Therefore, the tangent line is simply $-(x-1) = y - 1 \to y = -x + 2$
19.	E	$f'(x) = e^x - xe^{-x}, f''(x) = -2e^{-x} + xe^{-x}, f'''(x) = 3e^{-x} - xe^{-x}, f^{(n)}x = (-1)^n(xe^{-x} - ne^{-x}) = (-1)^n e^{-x}(x-n), n = 2022 \to (x-2022)e^{-x}$
20.	D	We see that $\frac{1}{x}$ and $\csc x$ both approach $+\infty$ when $x \to 0^+$ and negativity $-\infty$ when $x \to 0^-$. Therefore, $\lim_{x\to 0^+} = \infty \neq \lim_{x\to 0^-} = 0$, so limit DNE.
21.	A	Using L'Hospital rule: $\lim_{x\to 0^+}\frac{\tanh(x)}{\tan(x)} = \lim_{x\to 0^+}\frac{\operatorname{sech}^2 x}{\sec^2 x} = 1$
22.	A	The volume $LWH = V = (9-x)(9-x)x$. Therefore, setting the derivative to 0 gives: $V' = (9-x)^2 - 2(9-x)x = 0 \to x = 3$ or 9. Choosing $x = 3$ gives $V = 108$.
23.	E	Solve for when $x^{\frac{1}{2}} = -1 \pm \varepsilon$ gives $x = 1.01^2 = 1.0201$ or $x = 0.99^2 = 0.9801$ which is 0.0201 and 0.0199 away from 1. Thus, we choose the smaller value for δ: 0.0199.
24.	B	By plugging in x, y, z we simply get $3\cdot 1^2\cdot 3 + 1\cdot 2\cos(\pi\cdot 1 - \pi\cdot 3) = 11$
25.	E	We can rewrite this expression as $y = (2022^x)^{(2022^x)^{(2022^x)\cdots}} \to y = (2022^x)^y = 2022^{xy}$. Therefore, $y' = 2022^{xy}(x + xy')\ln 2022$. When $x = 0, y = 1$ so plugging these terms yields $y' = \ln 2022$.
26.	A	We can generically rewrite the integral as $I(m) = \int_{-\infty}^{\infty}\frac{\tan^{-1} mx}{x(1+x^2)}dx \to I'(m) = \int_{-\infty}^{\infty}\frac{1}{(1+m^2x^2)(1+x^2)}dx = \int_{-\infty}^{\infty}\frac{A}{1+m^2x^2} + \frac{B}{1+x^2}dx \to A + B = 1, A + Bm^2 = 0 \to A = \frac{-m^2}{1-m^2} B = \frac{1}{1-m^2} \to I'(m) = \frac{1}{1-m^2}\int_{-\infty}^{\infty}\frac{-m^2}{1+m^2x^2} + \frac{1}{1+x^2}dx = \frac{1}{1-m^2}[-m\tan^{-1} mx +$

		$\tan^{-1} x]_{-\infty}^{\infty} = \frac{-m\pi+\pi}{1-m^2} = \frac{\pi}{1+m} \to I(m) = \pi\ln(1+m) + C, I(0) = \int_{-\infty}^{\infty} \frac{\tan^{-1} 0}{x(1+x^2)} dx = 0 = \pi\ln(1+m) + C \to C = 0. I(2022) = \pi\ln(2023)$
27.	C	$\binom{b}{a-1} > \binom{b-1}{a} \to \frac{b!}{(a-1)!(b-a+1)!} > \frac{(b-1)!}{a!(b-a-1)!} \to \frac{\frac{b!}{(a-1)!(b-a+1)!}}{\frac{(b-1)!}{a!(b-a-1)!}} > 1 \to \frac{ab}{(b-a+1)(b-a)} > 1 \to$ $b^2 - 3ab + a^2 + b - a < 0$. The roots to this quadratic solving for b are $\frac{3a-1\pm\sqrt{(1-3a)^2-4(a^2-a)}}{2} = \frac{3a-1\pm\sqrt{5a^2-2a+1}}{2}$. Since we will select the larger root for $B(a)$, $\lim_{a\to\infty} \frac{B(a)}{a} = \lim_{a\to\infty} \frac{3a-1+\sqrt{5a^2-2a+1}}{2a} = \lim_{a\to\infty} \frac{3-\frac{1}{a}+\sqrt{5-\frac{2}{a}+1/a^2}}{2} = \frac{3+\sqrt{5}}{2}$
28.	B	$f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \csc x = \sin x + \cos x + \frac{1}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x}$. We can use the identity $\sin x + \cos x = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$ to give us $f(x) = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) + \frac{2}{\sin 2x} + \frac{2\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}{\sin 2x} = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) + \frac{2}{\cos\frac{\pi}{2}-2x} + \frac{2\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}{\cos\frac{\pi}{2}-2x} =$ $\sqrt{2}\cos\left(\frac{\pi}{4} - x\right) + \frac{2}{\cos 2\left(\frac{\pi}{4}-x\right)} + \frac{2\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}{\cos 2\left(\frac{\pi}{4}-x\right)} = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) + \frac{2}{2\cos^2\left(\frac{\pi}{4}-x\right)-1} + \frac{2\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}{2\cos^2\left(\frac{\pi}{4}-x\right)-1}$. We can set some variable $a = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$ such that $f(x) = a + \frac{2}{a^2-1} + \frac{2a}{a^2-1} \to$ $f(x) = a + \frac{2}{a-1}$. Taking the derivative with respect to a and setting it equal to 0 gives $a = 1 \pm \sqrt{2}$. Since $a = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$ then $a \in [-\sqrt{2}, \sqrt{2}]$. Plugging in $a = 1 - \sqrt{2}, \sqrt{2}$, $-\sqrt{2}$ results in $f(x) = 1 - 2\sqrt{2}, 2 - 3\sqrt{2}, 2 + 3\sqrt{2}$. For approximations use $\sqrt{2} \approx 1.4$ to see that $3\sqrt{2} - 2 > 2\sqrt{2} - 1$. Therefore, the minimum value of $\|f(x)\| = 2\sqrt{2} - 1$.
29.	A	Taking the derivative: $\left(\frac{P_n(x)}{(x^a-1)^{n+1}}\right)' = \left(\frac{P_n'(x)(x^a-1)^{n+1} - P_n(x)(ax^{a-1})(n+1)(x^a-1)^n}{(x^a-1)^{2(n+1)}}\right) =$ $\left(\frac{P_n'(x)(x^a-1) - P_n(x)(ax^{a-1})(n+1)}{(x^a-1)^{n+2}}\right) = \frac{P_{n+1}(x)}{(x^a-1)^{(n+1)+1}}$ which means that $P_{n+1}(x) = P_n'(x)(x^a - 1) - P_n(x)(ax^{a-1})(n+1)$. By substituting $x = 1$, we have $P_{n+1}(1) = P_n'(1)(1^a - 1) - P_n(1)(a1^{a-1})(n+1) = -a(n+1)P_n(1)$. Because $P_0(1) = 1$, we can observe that $P_1(1) = -a(0+1), P_2(1) = -a(2)(-a) = a^2 3!, P_3(1) = -a^3 3!,$ $P_4(1) = a^4 4!, \ldots, P_{2022}(1) = a^{2022} 2022!$
30.	B	Looking at the first few terms, we observe that $F_1(x) = x\ln x - x, F_2(x) = x\ln x - x, F_2(x) = \frac{1}{2}x^2\ln x - \frac{3}{4}x^2, F_3(x) = \frac{1}{6}x^3\ln x - \frac{11}{36}x^3$ to see that $F_n(x) = \frac{x^n}{n!}(\ln x - c_n)$. We can find a recurrence for c_n: $F_n'(x) = \frac{x^{n-1}}{(n-1)!}\ln x - \frac{x^{n-1}}{(n-1)!}c_n + \frac{x^{n-1}}{n!} =$

		$\frac{x^{n-1}}{(n-1)!}\ln x + \frac{x^{n-1}}{n!}(1 - nc_n) = \frac{x^{n-1}}{(n-1)!}(\ln x - c_{n-1}) \rightarrow \frac{x^{n-1}}{n!}(1 - nc_n) = \frac{x^{n-1}}{(n-1)!}(-c_{n-1}) \rightarrow$ $1 - nc_n = -nc_{n-1} \rightarrow c_n = c_{n-1} + \frac{1}{n}$. We know that $c_0 = 0$ so for $n > 0, c_n = \sum_{k=1}^{n} \frac{1}{k} \approx$ $\ln n + \gamma$, where γ is the Euler– Mascheroni constant ($\sim$0.577). Now, $\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n} = \lim_{n\to\infty} -\frac{c_n}{\ln n} = \lim_{n\to\infty} -\frac{\ln n + \gamma}{\ln n} = -1$