1. A
2. D
3. C
4. B
5. C
6. D
7. D
8. B
9. C
10. D
11. C
12. E
13. D
14. C
15. B
16. D
17. B
18. D
19. A
20. A
21. B
22. B
23. B
24. C
25. A
26. D
27. B
28. D
29. C
30. A

1. A The triangle is right, so the area is $\frac{1}{2}(6)(8) = 24$
2. D Using the substitution $u = \sin x$, the antiderivative is $\frac{1}{3}\sin^3 x$, so the integral is $\frac{1}{3}$.
3. C The factors are powers of 2 up to 128, which add to 255.
4. B The possible outcomes are 2, 4, 8, which have 1, 3, 5 possible ways to attain, respectively. That's 9 cases out of 36 possible, so the probability is $\frac{1}{4}$.
5. C $(3)(3) - (4i)(4i) = 25$
6. D By Vieta's, the sum is $-\frac{16}{2} = -8$
7. D The angle is coterminal with $\frac{4\pi}{3}$, thus the value is $-\frac{\sqrt{3}}{2}$.
8. B The positive root is $\frac{-b+\sqrt{b^2-4ac}}{2a}$, rationalizing the numerator, it is $-\frac{4ac}{2a(-b-\sqrt{b^2-4ac})}$, cancel out the $2a$ and setting the remaining a to 0 to get $r = \frac{c}{b}$.
9. C $f(x) = \left(\frac{x}{6}\right)^{\frac{3}{2}}$, then $f'(x) = \frac{3}{2}\left(\frac{x}{6}\right)^{\frac{1}{2}} \cdot \frac{1}{6} = \frac{1}{4}\left(\frac{x}{6}\right)^{\frac{1}{2}}$. Therefore, $f'(6) = \frac{1}{4}$, with $f(6) = 1$. So the approximation is $f(10) \approx 1 + \frac{1}{4}(10 - 6) = 2$.
10. D We can calculate the complement probability of Arnav the Ant never visiting the same edge twice, and then simply subtract 1 minus that probability for the desired answer. Consider the tetrahedron ABCD, with Arnav the Ant starting at A. WLOG, let's say his first move is to B. His next move has to be to either C or D, which gives probability of 2/3. WLOG, let's say he moves to C (the arrangement is symmetric). There's two cases from here. If his next move is to D, then his next steps must be to A and then to C. This gives probability of (1/3)(1/3)(1/3) = 1/27. If his next move is to A, then he must move to D next, and he has two options from there, either to B or C. This gives probability (1/3)(1/3)(2/3) = 2/27. Thus, the final probability is (2/3)(1/27 + 2/27) = 2/27. This gives the final answer 1 – 2/27 = 25/27.
11. C Note that if we take a vector $u = \langle 1,1,1 \rangle$, then $Pu = \langle a, b, c \rangle$, and $a + b + c$ will equal to the sum of the elements of P. Thus, we simply need to find the projection of u onto v to get $\frac{u \cdot v}{|v|^2} v = \langle \frac{1}{9}, -\frac{2}{9}, \frac{2}{9} \rangle$. The sum is then $\frac{1}{9}$.
12. E $P^2 u = P(Pu)$. In other words, $P^2 u$ projects the projection of u onto v onto v, which will not change. Therefore, $P^2 = P$, and the sum of the elements of $P^2 - P = 0$ is 0.
13. D For two circles of equal radius, the radical axis is exactly in the middle between the two circles, so it is the perpendicular bisector of the line segment joining the centers of the two circles. Therefore, the three radical axes in question are perpendicular bisectors of the sides of the triangle, and so the radical center is the circumcenter of the triangle.
14. C Let the three circles have radii x, y, z. Then we have x + y = c, x + z = b, y + z = a, where a, b, c are the side lengths of the triangle. Solving for x, y, z gives x = s – a, y = s – b, z = s – c. Also, for two circles that are tangent, the radical axis of the two circles will be the line that's tangent to both of those circles at their common tangency point. So, if we draw the three radical axes in question, we will see that they have to intersect at point, call it I, such that the three feet of I onto the sides of the triangle split the sides into lengths s – a & s – b, s – a & s – c, and s – b & s – c. However, if we draw the incircle of this triangle, draw the three radii tangent to the

sides, we will see that the tangency points of the incircle split the sides into the exact same pairs as we saw. Thus, I has to be the incenter.

15. B Note that the generating function $f(x) = \sum_{n=2}^{\infty} a_n x^n$. Note the indexing starts at 2, as the first two terms are 0. We now have

$$f(x) = \sum_{n=2}^{\infty}(2a_{n-1} - a_{n-2} + 1)x^n = \sum_{n=2}^{\infty} 2a_{n-1}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} x^n$$

We can now express the first two sums in terms of $f(x)$ by adjusting the powers of x to match the index on a:

$$f(x) = \sum_{n=2}^{\infty} 2a_{n-1}x^{n-1} \cdot x - \sum_{n=2}^{\infty} a_{n-2}x^{n-2} \cdot x^2 + \sum_{n=2}^{\infty} x^n$$

The third term is an infinite geometric sequence. We now have

$$f(x) = 2xf(x) - x^2 f(x) + \frac{x^2}{1-x}$$

Solving for $f(x)$,

$$(1 - 2x + x^2)f(x) = \frac{x^2}{1-x}$$

$$f(x) = \frac{x^2}{(1-x)^3}$$

Thus $f\left(\frac{1}{3}\right) = \frac{1/9}{8/27} = \frac{3}{8}$.

16. D Note that all probability must sum to 1, so $f(1) = 1$. Next, we wish to find the expected value, which is $\sum_{n=1}^{\infty} na_n$.
Also note $f'(x) = \sum_{n=1}^{\infty} nx^{n-1}a_n$. We can see that $f'(1)$ is equal to the expected value.

17. B We are given that $\int_0^{10} |f(x)|\, dx = 10$. We can apply Cauchy-Schwarz Inequality using $g(x) = 1$:

$$10^2 = \left(\int_0^{10} |f(x)|\, dx\right)^2 \leq \int_0^{10} |f(x)|^2\, dx \int_0^{10} 1^2\, dx = 10 \int_0^{10} |f(x)|^2\, dx$$

Therefore, $\pi \int_0^{10} |f(x)|^2\, dx \geq 10\pi$

18. D We are given that $\pi \int_0^{3} |f(x)|^2\, dx$. Let $g(x) = x$ and applying Cauchy-Schwarz Inequality, we have

$$\left(\int_0^{3} xf(x)\, dx\right)^2 \leq \int_0^{3} |f(x)|^2\, dx \int_0^{3} x^2\, dx = 81$$

Therefore, $2\pi \int_0^{3} xf(x)\, dx \leq 18\pi$

19. A Exchange n in the given theorem with $n \ln n$, we have

$$\lim_{n\to\infty} \frac{\pi(n \ln n)}{n \ln n \,/\, \ln(n\, ln\, n)} = 1$$

Since

$$\lim_{n\to\infty} \frac{\pi(n \ln n)}{f(n)} = 1$$

We also have

$$\lim_{n\to\infty}\frac{f(n)}{n\ln n\,/\ln(n\,ln\,n)}=\lim_{n\to\infty}\frac{f(n)(\ln n+\ln(ln\,n))}{n\ln n}=1$$

This can be split into

$$\lim_{n\to\infty}\frac{f(n)}{n}\cdot\lim_{n\to\infty}\frac{\ln n+\ln(ln\,n)}{\ln n}=\lim_{n\to\infty}\frac{f(n)}{n}=1$$

20. A Use the given theorem, the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ has the same convergence as A using the Limit Comparison Test. This series converges by the integral test, so A converges. For B, note that $\pi(p_n)=n$, using the limit from the previous question, we have

$$\lim_{n\to\infty}\frac{\pi(n\ln n)}{\pi(p_n)}=1$$

Thus we can approximate p_n with $n\ln n$. Using this, we see that the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges by the integral test, and so B must converge as well.

21. B Let's say that the midsegment tangent to the incircle is parallel to the side with length a, and the other two sides have length b and c. We see that incircle has a quadrilateral in which it's inscribed. For a quadrilateral with an inscribed circle, the lengths of its opposite sides must sum to the same value. Therefore, we can say that a / 2 + a = b / 2 + c=2, or equivalently, 3a = b + c. There are several cases. If a = 11, then the third side has length 27, which does not produce an actual triangle. If a = 6, then the third side has length 7, which works. If b and c happen to be 6 and 11, then the third side is not an integer, and so the third side must have length 7.

22. B Note that Conhur's sequence of coin flips will have exactly seven arms and tails flips, while the other three will always be heads. Thus, we can simplify the problem by just looking at the sequence that excludes the heads, because the location of the heads in the sequence does not affect the probability we have to calculate. For the sequence of 7 coin flips, we can calculate in how many cases there's less than 2 arms. We can do this by finding $\binom{7}{2}+\binom{7}{1}+\binom{7}{0}=29$. With $2^7=128$ total possible outcomes, the desired probability is $\frac{99}{128}$.

23. B Placing the triangle onto the coordinate plane with B at the origin, C at $(14,0)$, and A at $(9,12)$. We can then find $O=\left(7,\frac{33}{8}\right)$ and $I=(8,4)$. Applying Shoelace Formula, we find that the area of ΔBOI IS $\frac{5}{2}$

24. C Note that if we divide the expression by 1000^{2020}, we can rewrite it as $\left(1+\frac{1}{1000}\right)^{1000\cdot 2.02}$, this expression is approximately to $e^{2.02}$. e^2 is slightly larger than 7, so the first digit must be 7.

25. A Since they all must cover the exact same distance to reach the same point as the same time, their meeting point is the circumcenter of the triangle with their initial locations as vertices. Using the formula for circumcenter, we find that the distance they cover is $R=\frac{abc}{4[ABC]}=\frac{17\cdot 28\cdot 25}{4\sqrt{35\cdot 7\cdot 18\cdot 10}}=\frac{85}{6}$.

26. D Note that for an absolute unit a, $\overline{a}=\frac{1}{a}$. Thus if we take the complex conjugate of the first equation, we will get $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1-i$. Multiply this by abc to get

$$ab + bc + ca = \frac{7}{5} + \frac{1}{5}i$$

Therefore,

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 2i - \left(\frac{14}{5} + \frac{2}{5}i\right) = -\frac{14}{5} + \frac{8}{5}i$$

27. B Since the two angles are decreasing and increasing at the same rate, the third angle stays the same. Note that at the given instant, the triangle is a right triangle, and the angle that stays constant is a right angle. Thus, we see that A actually moves along a circle centered at the midpoint of BC. Thus, at the given instant, we can express A's movement as a vector. Note that the central angle of A along the circle changes at a rate of 2 radians per seconds, and so the magnitude of the vector in question is $r\theta' = 2$ units per second. At the given instant, the direction of the vector can be found to be $\frac{\pi}{6}$. If we consider BC to be the 0 radian direction. Thus, we can find that A is moving away from BC at a rate of $2\sin\frac{\pi}{6} = 1$ unit per second. This is the rate at which the altitude of the triangle is changing. The area is changing at a rate of $\frac{bh'}{2} = \frac{2\cdot 1}{2} = 1$ unit squared per second.

28. D Let $E(n)$ be the expected number of seconds for Kev to reach position $x = 4$. Note that $E(4) = 0$, and $E(n) = 3$ for $n > 4$, as it is possible for Kev to skip 4, and the only way to get back to 4 is by teleporting. That is a geometric distribution, with an expected value of 3. Now the task is to compute $E(0)$.
When Kev is at a position n less than 4, a move must be taken. Then Kev lands on $n + 1, n + 2, 4$ with equal probability (with the possibility of $n + 1$ or $n + 2$ being equal to 4 as well). Therefore,

$$E(3) = 1 + \frac{1}{3}\big(E(4) + E(5) + E(4)\big) = 1 + \frac{1}{3}(0 + 3 + 0) = 2$$
$$E(2) = 1 + \frac{1}{3}\big(E(3) + E(4) + E(4)\big) = 1 + \frac{1}{3}(2 + 0 + 0) = \frac{5}{3}$$
$$E(1) = 1 + \frac{1}{3}\big(E(2) + E(3) + E(4)\big) = 1 + \frac{1}{3}\left(\frac{5}{3} + 2 + 0\right) = \frac{20}{9}$$
$$E(0) = 1 + \frac{1}{3}\big(E(1) + E(2) + E(4)\big) = 1 + \frac{1}{3}\left(\frac{20}{9} + \frac{5}{3} + 0\right) = \frac{62}{27}$$

29. C Expanding the equation to get

$$a^4 + 4b^4 = 1 + 4a^2b^2 \rightarrow (a^2 - 2b^2)^2 = 1 \rightarrow a^2 - 2b^2 = \pm 1$$

This is a Pell equation. The solution pairs can be found using either convergents of $\sqrt{2}$ or with $a + b\sqrt{2} = \left(1 + \sqrt{2}\right)^n$. The fourth smallest solution pair occur at $a = 17, b = 12$. Thus $a + b = 29$.

30. A We can find the area of $ABCD$ to be $\frac{1}{2}(20)(18)\sin 60° = 90\sqrt{3}$. Let $QRST$ be the quadrilateral formed by connecting the four midpoints of the sides of $ABCD$. Then $[QRST] = \frac{1}{2}[ABCD] = 45\sqrt{3}$. $WXYZ$ is simply $QRST$ dilated by a factor of $\frac{2}{3}$ with respect to P, as the centroid of a triangle is $\frac{2}{3}$ of the way from a vertex to the opposite midpoint. Therefore $[WXYZ] = \left(\frac{2}{3}\right)^2 [QRST] = 20\sqrt{3}$