SOLUTIONS

1. B
2. D
3. C
4. B
5. C
6. A
7. A
8. C
9. B
10. E
11. D
12. B
13. C
14. B
15. A
16. D
17. A
18. C
19. C
20. D
21. E
22. B
23. A
24. D
25. C
26. B
27. C
28. D
29. B
30. B

1. $\lim_{z \to 2-3i} (z^2 - |z| + 1) = (4 - 3i)^2 - \sqrt{(4)^2 + (-3)^2} + 1$

$\lim_{z \to 2-3i} (z^2 - |z| + 1) = (16 - 24i - 9) - 5 + 1$

$\lim_{z \to 2-3i} (z^2 - |z| + 1) = 3 - 24i$

Hence the answer is B

2. $\ln z = \ln(e^a e^{bi})$

$\ln z = \ln(e^a) + \ln(e^{bi})$

$\ln z = \ln(e^a) + bi$

$\tan\theta = \frac{b}{a} = \sqrt{3}$

$\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$|2\sqrt{3} + 6i| = 4\sqrt{3}$

$\ln(2\sqrt{3} + 6i) = \ln(4\sqrt{3}) + \frac{\pi i}{3}$

Hence the answer is D

3. $\frac{1}{2}|z| = |z - 3|$

$\frac{1}{2}\sqrt{x^2 + y^2} = \sqrt{(x-3)^2 + y^2}$

$\frac{1}{4}(x^2 + y^2) = x^2 - 6x + 9 + y^2$

$x^2 + y^2 = 4x^2 - 24x + 36 + 4y^2$

$0 = 3x^2 - 24x + 36 + 3y^2$

$0 = x^2 - 8x + 12 + y^2$

$-12 = (x^2 - 8x) + y^2$

$-12 + 16 = (x - 4)^2 + y^2$

$(x - 6)^2 + y^2 = 4$

$A = 4\pi$

Hence the answer is C

4. $\frac{\text{cis } 195°}{\text{cis } 75°} = \text{cis}(195° - 75°) = \text{cis}(120°)$

$\frac{\text{cis } 195°}{\text{cis } 75°} = \text{cis}(120°)$

$\frac{\text{cis } 195°}{\text{cis } 75°} = \frac{i\sqrt{3}-1}{2}$

Hence the answer is B

5. $e^{\frac{7\pi}{12}i} = \text{cis}\left(\frac{7\pi}{12}\right)$

$e^{\frac{7\pi}{12}i} = \text{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$Im\left(e^{\frac{7\pi}{12}i}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$Im\left(e^{\frac{7\pi}{12}i}\right) = \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

$Im\left(e^{\frac{7\pi}{12}i}\right) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$Im\left(e^{\frac{7\pi}{12}i}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

Hence the answer is C

6. By definition $\{2, i, 2i\}$ is a solution set. Hence $f(-i) = f(-2i) = 0$ because complex roots occur in pairs

Let $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

$f(x) = (x-2)(x^2+1)(x^2+4)$

$f(x) = x^5 - 2x^4 + 5x^3 - 10x^2 + 4x - 8$

$a + b + c + d + e + f = 1 - 2 + 5 - 10 + 4 - 8 = -10$

Hence the answer is A

7. $\sqrt{-6} \times \sqrt{-4} = i\sqrt{6} \cdot i\sqrt{4}$

$\sqrt{-6} \times \sqrt{-4} = -2\sqrt{6}$

Hence the answer is A

8. The n^{th} roots of complex numbers are denoted as follows:

$$(rcis\ \theta)^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \text{ where } k = 0, 1, 2, 3, 4, \dots, n-1$$

This gives *n* complex n^{th} roots of unity. All the roots have a magnitude of 1.

Each root lies on the unit circle and the angle between any two consecutive roots is $\frac{2\pi}{n}$

The roots are evenly spaced around the unit circle.

$\frac{2\pi}{n} = \frac{360°}{40} = 9°$

The roots are $0°, 9°, 18°, 27°, \dots, 9(40-1)°$

The roots in quadrant II are $99°, 108°, \dots 171°$ because points on axes do not lie in any quadrant

$99 = 9(12-1)$ and $261 = 9(20-1)$

There are roots for $k = 12, 13, 14, \dots, 20$ or 9 roots in Quadrant II

Hence the answer is C

9. $|z_1 z_2 z_3 z_4| = |z_1||z_2||z_3||z_4|$

$|(6+2i)(3+4i)(4+8i)(1-2i)| = |(6+2i)||(3+4i)||(4+8i)||(1-2i)|$

$|(6+2i)(3+4i)(4+8i)(1-2i)| = (\sqrt{40})(\sqrt{25})(\sqrt{80})(\sqrt{5})$

$|(6+2i)(3+4i)(4+8i)(1-2i)| = (2\sqrt{10})(5)(4\sqrt{5})(\sqrt{5})$

$|(6+2i)(3+4i)(4+8i)(1-2i)| = (10\sqrt{10})(20)$

$|(6+2i)(3+4i)(4+8i)(1-2i)| = 200\sqrt{10}$

Hence the answer is B

10. $(2\sqrt{3}-2i)^8 = 4^8\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)^8$

$(2\sqrt{3}-2i)^8 = 4^8\left(\text{cis} - \frac{\pi}{6}\right)^8$

$(2\sqrt{3}-2i)^8 = 4^8 \text{ cis} - \frac{4\pi}{3}$

$(2\sqrt{3}-2i)^8 = 4^8 \text{ cis} \frac{2\pi}{3}$

$(2\sqrt{3}-2i)^8 = 4^8\left(\frac{-1+i\sqrt{3}}{2}\right) = 2^{15}(-1+i\sqrt{3})$

Hence the answer is E

11. $\begin{bmatrix} i & 3 \\ -2i & 4 \end{bmatrix}\begin{bmatrix} -3i & 4i & -i \\ 2 & i & 3 \end{bmatrix} = \begin{bmatrix} 3+6 & -4+3i & 1+9 \\ -6+8 & 8+4i & -2+12 \end{bmatrix}$

$\begin{bmatrix} i & 3 \\ -2i & 4 \end{bmatrix}\begin{bmatrix} -3i & 4i & -i \\ 2 & i & 3 \end{bmatrix} = \begin{bmatrix} 9 & 3i-4 & 10 \\ 2 & 8+4i & 10 \end{bmatrix}$

Hence the answer is D

12. $d = \sqrt{(6-2)^2 + (-5-3)^2} = \sqrt{80} = 4\sqrt{5}$

Hence the answer is B

13. This is analogous to the point $(2,2i)$ and the origin; translating the graph left 2 units and down i unit to obtain $(0, i)$ and $(-2, -i)$ respectively.

$(2+2i)[cis(60°)] = (2+2i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$(2+2i)[cis(60°)] = 1 + i\sqrt{3} + i - \sqrt{3}$

$(2+2i)[cis(60°)] = (1-\sqrt{3}) + i(1+\sqrt{3})$

Translating the point 2 units left and i unit down → $(1-\sqrt{3}) + i(1+\sqrt{3})$

Hence the answer is C

14. The determinant of a triangular matrix is the product along the main diagonal

$D = (3)(4i+1)(1+2i)$

$D = (3)(4i - 8 + 1 + 2i)$

$D = 18i - 21$

Hence the answer is B

15. $6 + 0i$ and $0 + 8i$ are the foci of the ellipse, so $2c = 10$. We also have $2a = 26$. Thus the eccentricity is $\frac{5}{13}$.

Hence the answer is A

16. From above, we can see that $b = 12$. So the area is $\pi ab = 156\pi$

Hence the answer is D

17. $(\text{cis } 25°)(2 \text{ cis } 95°)(\text{cis } 210°) = 2 \text{ cis } 330°$

$e^{a+bi} = e^a \text{ cis } b$

$e^a = 2 = e^{ln2}$

$b = -\frac{\pi}{6}$

$(\text{cis } 25°)(2 \text{ cis } 95°)(\text{cis } 210°) = e^{ln2 - \frac{11\pi}{6}i}$

Hence the answer is A

18. I is not true because the logarithm of z is in base 10 and not base e

II is true because $|z|^2 = z\bar{z}$

III is the parallelogram law and true

Hence the answer is C

19. $f(x) = x^6 - 4x^4 + 3x^2 - 12$

$f(x) = (x^4)(x^2 - 4) + (3)(x^2 - 4)$

$f(x) = (x^4 + 3)(x^2 - 4)$

There are four nonreal roots

Hence the answer is C

20. Let $z = a + bi$, then

$z\bar{z} = a^2 + b^2$ and

$Im(z^2) = Im(a^2 + 2abi - b^2) = 2ab$

$a^2 + b^2 = \frac{1}{2}$ and $2ab = \frac{\sqrt{2}}{3}$

$ab = \frac{1}{3\sqrt{2}} = \frac{1}{\sqrt{18}}$

Factors of 18 are 1, 2, 3, 6, 9, 18

$\left(\frac{1}{\sqrt{1}}\right)^2 + \left(\frac{1}{\sqrt{18}}\right)^2 \neq \frac{1}{2}$

$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{9}}\right)^2 \neq \frac{1}{2}$

$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{1}{2}$

$z = \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}}$

$\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} = \frac{\sqrt{6}-2\sqrt{3}}{6}$

Hence the answer is D

21. $z_a\overline{z_b} - \overline{z_a z_b} = (1 + 5i)(2i - 6) + \overline{(1 + 5i)(-6 - 2i)}$

$z_a\overline{z_b} - \overline{z_a z_b} = (2i - 6 - 10 - 30i) + (\overline{-6 - 2i - 30i + 10})$

$z_a\overline{z_b} - \overline{z_a z_b} = (-28i - 16) + (\overline{4 - 32i})$

$z_a\overline{z_b} - \overline{z_a z_b} = (-16 - 28i) + (4 + 32i)$

$z_a\overline{z_b} - \overline{z_a z_b} = -12 + 4i$

This is in Quadrant 2, so the argument is $\pi - \arctan\left(\frac{1}{3}\right)$

Hence the answer is E

22. $\overline{z_a} - z_b = (1 - 5i) + (-6 - 2i) = -5 - 7i$

$|\overline{z_a} - z_b| = \sqrt{25 + 49} = \sqrt{74}$

Hence the answer is B

23. $z_a + \overline{z_a z_b} = 1 + 5i - \overline{(1 - 5\imath)(-6 - 2\imath)}$

$z_a + \overline{z_a z_b} = 1 + 5i - \overline{(-6 - 2\imath + 30\imath - 10)}$

$z_a + \overline{z_a z_b} = 1 + 5i - \overline{-16 + 28\imath}$

$z_a + \overline{z_a z_b} = 1 + 5i - (-16 - 28i) = 17 + 33i$

Hence the answer is A

24. $e^{\left(3ln2 + i\frac{4\pi}{3}\right)} = 8e^{i\frac{4\pi}{3}}$

$e^{\left(3ln2 + i\frac{4\pi}{3}\right)} = 8\,\mathrm{cis}(240°)$

$e^{\left(3ln2 + i\frac{4\pi}{3}\right)} = 8\left(-\frac{1 + i\sqrt{3}}{2}\right) = -4 - 4i\sqrt{3}$

Hence the answer is D

25. $S = \frac{a_1}{(1 - r)}$ given $a_n = a_1 r^{(n-1)}$

$S = \frac{2}{1 - \left(-\frac{2}{3}i\right)}$

$S = \frac{6}{3 + 2i}$

$S = \frac{6}{3 + 2i}\left(\frac{3 - 2i}{3 - 2i}\right) = \frac{18 - 12i}{13}$

$Im(S) = -\frac{12}{13}$

Hence the answer is C

26. The n^{th} roots of complex numbers are denoted as follows:

$$(rcis\ \theta)^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$ where $k = 0, 1, 2, 3, 4, \dots, n - 1$

This gives *n* complex n^{th} roots of unity. All the roots have a magnitude of 1.

Each root lies on the unit circle and the angle between any two consecutive roots is $\frac{2\pi}{n}$

The roots are evenly spaced around the unit circle.

$cis\left(15(x°)\right) = \left(cis\,(x°)\right)^{15}$

$\left(cis\,(x°)\right)^{15} = -1$

There are 15 evenly spaced roots beginning with the principle root $x = \frac{\pi+2(0)\pi}{n} = \frac{180°+0}{15} = 12°$

$\frac{2\pi}{n} = \frac{360°}{15} = 24°$

The roots in Quadrant I are $12°, 36°, 60°, 84°$

Hence the answer is B

27. Since the distance between $(6+0i)$ and $(0+8i)$ is 10. The equality is only satisfied when z lies on the segment between those 2 points.

Hence the answer is C

28. $\frac{7+5i}{2-i} = \frac{7+5i}{2-i}\left(\frac{2+i}{2+i}\right) = \frac{14+7i+10i-5}{4+1}$

$\frac{7+5i}{2-i} = \frac{9+17i}{5}$

Hence the answer is D

29. Let $x = \cfrac{5}{6i+\cfrac{5}{6i+\cfrac{5}{6i+\cdots}}}$

$x = \frac{5}{6i+x}$

$x^2 + 6ix = 5$

$x^2 + 6ix - 5 = 0$

$(x+i)(x+5i) = 0$

$x = -i$ and $x = -5i$

So all possible values of x^2 are -1 and -25.

Hence the answer is B

30. $i^{2020} = (-1)^{1010} = 1$

Hence the answer is B