Alpha

Sequences and Series

Test #622

Directions:

1. Fill out the top section of the Round 3 Google Form answer sheet and select **Alpha-Sequences and Series** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

For all question, NOTA denotes "None of the above".

1. What is units digit of the sum $1 + 2 + 3 + \cdots + 2022$?

   A. 0 B. 1 C. 2 D. 3 E. NOTA

2. Suppose that a_n is an arithmetic sequence. If $\frac{a_5}{a_3} = 2$, what is the value of $\frac{a_4}{a_2}$?

   A. 1 B. 2 C. 3 D. 4 E. NOTA

3. How many terms are there in the sequence $6, 13, 20, 27, \cdots, 2022$?

   A. 288 B. 289 C. 290 D. 291 E. NOTA

4. The sequence $a_1, a_2, \cdots, a_n$ consists of n consecutive positive integers whose sum is equal to 2022. What is the largest possible value of n?

   A. 3 B. 4 C. 12. D. 337 E. NOTA

5. If $1 + 2x + 3x^2 + 4x^3 + \cdots = 25$, what is x?

   A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{3}{4}$ D. $\frac{4}{5}$ E. NOTA

6. Suppose that the sequence a_n satisfies $na_{n+1} = 2(n+1)a_n, a_1 = 1$. Find the units digit of a_{101}.

   A. 6 B. 3 C. 2 C. 1 E. NOTA

7. How many positive integers less than 2022 can be expressed as a sum of 7 consecutive positive multiples of 13?

A. 22 B. 21 C. 20 D. 19 E. NOTA

8. If a_n is a geometric sequence of positive real numbers with $a_3 = 12$ and $a_9 = 75$, find the value of a_6.

A. 18 B. 24 C. 30 D. 36 E. NOTA

9. Suppose that a sequence a_n satisfies $a_1 + a_2 + \cdots + a_n = n! + (n+1)!$. Find the remainder when a_{19} is divided by 19.

A. 1 B. 2 C. 18 D. 0 E. NOTA

10. Let b_n be a sequence defined by $b_{n+2} = b_{n+1}b_n$ with $b_0 = 1, b_1 = 2$. Find the units digit of b_{11}.

A. 8 B. 6 C. 4 D. 2 E. NOTA

11. Let a_n be a sequence defined by $a_n = n^2 + 1$. If $a_{11} \cdot a_{12} \cdot a_{13} \cdot a_{14} = a_p \cdot a_q$,

what is $p + q$?

A. 316 B. 320 C. 423 D. 515 E. NOTA

12. If the three sides of a right triangle form an arithmetic sequence, and the perimeter of the triangle is 60, find the length of the hypothenuse.

A. 24 B. 25 C. 26 D. 30 E. NOTA

13. Suppose that a_n is an arithmetic sequence and $a_1 + a_2 + \cdots + a_{99} = 9999$. What is the value of a_{50}?

A. 99 B. 100 C. 101 D. 102 E. NOTA

14. When the following sum S is evaluated, find the hundreds digit of S.

$$S = 40^2 + 39^2 - 38^2 - 37^2 + 36^2 + 35^2 - 34^2 - 33^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$$

A. 8 B. 6 C. 5 D. 3 E. NOTA

15. If an arithmetic sequence a_n with $a_1 \neq 0$ satisfies $a_{11} + a_{12} + \cdots + a_{20} = 5(a_1 + a_2 + \cdots + a_{10})$, what is $\frac{a_1}{a_2}$?

A. 2 B. -2 C. $-\frac{1}{2}$ D. 5 E. NOTA

16. Suppose that the three sides of a right triangle form a geometric sequence. If the length of the shorter leg is 2, what is the length of the hypotenuse?

A. $\sqrt{3}$ B. $\sqrt{3}+1$ C. $\sqrt{5}+1$ D. 4 E. NOTA.

17. Find the value of the following sum: $i + i^2 + \cdots + i^{2022}$.

A. 0 B. 1 C. i D. -1 E. NOTA.

18. Consider a sequence defined as $a_{n+1} = (a_n)^2$ for $n \geq 1$. If $a_1 \neq a_2$ and $a_1 = a_4$, then a_1 can be expressed as $e^{\frac{m\pi}{n}i}$ for relatively prime positive integers m and n. Find the smallest possible value of $m + n$.

A. 7 B. 8 C. 9 D. 10 E. NOTA

19. The five interior angles of a convex pentagon form an arithmetic sequence of positive integers. Find the smallest possible value of the smallest interior angle the pentagon.

A. 35 B. 38 C. 41 D. 42 E. NOTA

20. The first and eleventh terms of an arithmetic sequence are $\frac{3}{7}$ and $\frac{2}{3}$. If the sixth term of the sequence is written as $\frac{p}{q}$ where p and q are relatively prime positive integers, what is the value of $p+q$?

A. 45 B. 65 C. 73 D. 89 E. NOTA

21. Let L_n be the Lucas sequence defined by $L_n = L_{n-1} + L_{n-2}; L_0 = 2, L_1 = 1$.
Find the infinite sum:

$$L_0 + \frac{L_1}{2} + \frac{L_2}{4} + \frac{L_3}{8} + \cdots$$

A. 3 B. 4 C. 5 D. 6 E. NOTA

22. Alicia's birthday is on January 15, and her two siblings', Brett and Clair's, birthdays come later in the year in that order. One day, Alicia calculated the number of days between two neighboring birthdays in a non-leap year cycle: from Alicia's birthday to Brett's birthday, Brett's birthday to Clair's birthday, and Clair's birthday to Alicia's next birthday. (Alicia counted the number of days by excluding the first day and including the last day. For example, there are 12 days from January 15 to January 27.) Then she learned that the three numbers form an increasing geometric sequence. How many days are there from Clair's birthday to Alicia's birthday?

A. 180 B. 225 C. 300 D. 320 E. NOTA

23. Suppose that $S_n = n^3$ represents the sum of first n terms of the sequence a_n. Find the units digit of a_{2022}.

A. 1 B. 3 C. 7 D. 9 E. NOTA

24. Find the infinite sum: $\sum_{n=1}^{\infty} \frac{2n}{n^4+n^2+1}$.

A. 1 B. 2 C. 3 D. 4 E. NOTA

25. The Fibonacci sequence F_n is defined by $\mathrm{F_n} = \mathrm{F_{n-1}} + \mathrm{F_{n-2}}; \mathrm{F_0} = 0, \mathrm{F_1} = 1$. Given that $F_{11} = 89$ and $F_{13} = 233$, find the number of positive divisors of $F_{13}^3 - F_{12}^3 - F_{11}^3$.

A. 60 B. 64 C. 72 D. 80 E. NOTA

26. Suppose that the recursive relation $a_{n+4} = a_{n+2} - a_n$ is true for all $n = 1,2,3,\cdots$ where $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$. What is the value of a_{2022} ?

A. 1 B. 2 C. 3 D. 4 E. NOTA

27. If we define a function $f(x)$ by $f(x) = \frac{2}{4^x+2}$, what is the value of the sum

$$f\left(\frac{1}{2022}\right) + f\left(\frac{2}{2022}\right) + f\left(\frac{3}{2022}\right) + \cdots + f\left(\frac{2021}{2022}\right) ?$$

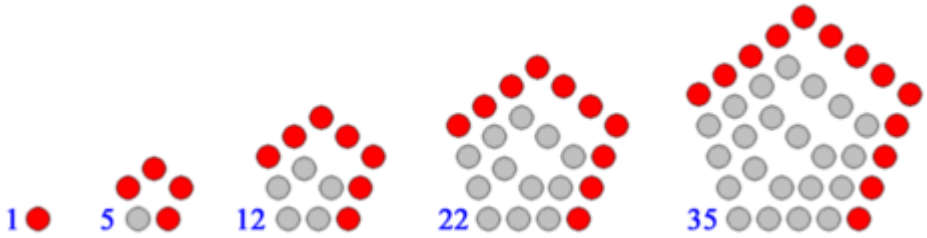
A. 1011 B. 1010 C. 1009 D. 1008 E. NOTA

28. When the following sum S is evaluated, what is the tens digit of the sum?

$$S = 101^3 - 100^3 + 99^3 - 98^3 + 97^3 - 96^3 + \cdots - 2^3 + 1^3$$

A. 0 B. 1 C. D. 3 E. 9 E. NOTA

29. The first five terms of pentagonal numbers P_n are $1, 5, 12, 22, 35$ as shown in the figure. What is the 20th pentagonal number P_{20}?



A. 532 B. 590 C. 651 D. 750 E. NOTA

30. Find the smallest positive integer $n > 1$ such that n-th pentagonal number P_n is a perfect square.

A. 49 B. 63 C. 81 D. 99 E. NOTA