Alpha

Analytic Geometry

Test #621

Directions:

1. Fill out the top section of the Round 3 Google Form answer sheet and select **Alpha-Analytic Geometry** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Unless otherwise specified, the domain and range of functions are limited to the real numbers. The bolded characters **i**, **j**, and **k** refer to the unit vectors $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, and $\langle 0,0,1\rangle$, respectively. "NOTA" stands for "None Of These Answers." Good luck!

1. Ann gives a point (on a sheet of paper) and a length l to Drew, who then draws the locus of points (on the sheet of paper) located a distance of l away from Ann's point. Which correctly describes the shape Drew drew?

   A. Circle B. Line C. Parabola D. Point E. NOTA

2. Triangle ABC has vertices $A(0,2)$, $B(1,0)$, and $C(3,0)$. Find the volume of the figure formed by revolving triangle ABC completely around the y-axis.

   A. $32/3$ B. $16\pi/3$ C. $8\pi/3$ D. 6π E. NOTA

3. Find an equation of the plane defined by vectors $\langle 2,3,-1\rangle$ and $\langle -4,1,3\rangle$ and passing through the point $(6,4,12)$.

   A. $3x+2y+6z=110$ B. $3x+2y+6z=-4$
   C. $5x-y+7z=110$ D. $5x+y+7z=118$ E. NOTA

4. Complex numbers $A=6\cos(\frac{2\pi}{3})+6i\sin(\frac{2\pi}{3})$ and $B=3\cos(\frac{5\pi}{3})+3i\sin(\frac{5\pi}{3})$, where $i=\sqrt{-1}$, are graphed as points in the Argand plane. Find the distance $\overline{AB}$.

   A. 3 B. $3\sqrt{5}$ C. $3\sqrt{7}$ D. 9 E. NOTA

5. Which of the following is an equation of the directrix of $2x^2+8x-3y+20=0$?

   A. $x=-\frac{19}{8}$ B. $x=-\frac{13}{6}$ C. $y=\frac{35}{8}$ D. $y=\frac{5}{2}$ E. NOTA

6. Point $P(4,-6)$ is rotated $30°$ counterclockwise about the origin to point $P'(a,b)$. What is $a+b$?

A. $-1-\sqrt{3}$ B. $\frac{-3+\sqrt{3}}{2}$ C. $-5-\sqrt{3}$ D. $5-\sqrt{3}$ E. NOTA

7. An ellipse has a focus at the origin. Its directrix nearest to this focus has equation $y=\frac{\sqrt{21}}{2}x-\frac{25}{2}$, and the endpoint of the major axis nearest to this focus is $(\frac{\sqrt{21}}{5},-\frac{2}{5})$. What is the eccentricity of the ellipse?

A. $3/4$ B. $1/4$ C. $1/5$ D. $1/20$ E. NOTA

8. Two vertices of a cube are at the points $(-1,4,5)$ and $(6,5,3)$. Find the geometric mean of all possible values of the volume of the cube.

A. $162\sqrt[3]{36}$ B. $162\sqrt{6}$ C. 162 D. $81\sqrt[4]{72}$ E. NOTA

9. Convert $y=x\tan(2x^2+y^2)$ to polar coordinates, for θ where defined.

A. $r=\sqrt{\frac{\theta}{1+(\cos\theta)^2}}$ B. $r=\pm\sqrt{\frac{\theta}{1+(\cos\theta)^2}}$

C. $r=\sqrt{\frac{\theta}{1+(\sin\theta)^2}}$ D. $r=\frac{\theta}{1+(\sin\theta)^2}$ E. NOTA

10. Three vertices of a parallelogram are $(-8,-2)$, $(2,-5)$, and $(-3,-11)$. Let $\{A_1,A_2,\ldots,A_n\}$ be the set of the n possible locations for the fourth vertex of the parallelogram. What is the area of the convex polygon $A_1A_2\ldots A_n$?

A. 300 B. 150 C. 75 D. $75/2$ E. NOTA

11. Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have magnitudes 6 and $2\sqrt{7}$, respectively. If the vector projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$ is $\langle 1, 3, -2\rangle$, find the sine of the angle $\theta \leq 180°$ between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

A. $\frac{\sqrt{29}}{6}$ B. $\frac{\sqrt{2}}{2}$ C. $\frac{\sqrt{14}}{6}$ D. $\frac{\sqrt{22}}{6}$ E. NOTA

12. Find the distance from the center of the conic $25(x-8)^2 - 144(y+17)^2 = 3600$ to one of its foci.

A. 5 B. 12 C. 13 D. 15 E. NOTA

13. Which of the following **always** define a unique plane?

I) 3 distinct points
II) 2 distinct lines
III) 2 distinct vectors and a point
IV) A line l and a point not on l

A. Only I, II, and IV B. Only IV C. Only III and IV D. Only II and III E. NOTA

14. Circles O and P, centered at $(-6, 3)$ and $(9, -17)$ respectively, have an interior tangent of length 24. What is the shortest distance between a point on P and a point on O?

A. 18 B. 20 C. 22 D. 24 E. NOTA

15. A point z in the Argand plane satisfies the equation $4(z+\overline{z}\,)^2 - (z-\overline{z}-20i)^2 = 144$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. What is the minimum value of $|z|^2$?

A. 45 B. 109 C. 9 D. 16 E. NOTA

16. Find the area enclosed by the graph of $|3x-2| + |7y+41| = 63$.

A. 189 B. 522 C. 756 D. 7938 E. NOTA

17. Senaye walks a path in the shape of the graph of $r = 2 + 3\cos\theta$. When her path is placed on the Cartesian plane, what is Senaye's distance from the x-axis when her x-coordinate is at its minimum?

A. $\frac{2\sqrt{2}}{3}$ B. $\frac{2}{3}$ C. 1 D. 0 E. NOTA

18. In the 3rd century B.C.E., Archimedes showed that the area of a parabolic sector is given by $\frac{2}{3}Bh$, where B and h are the base and height of the sector, respectively. What is the area enclosed by the graphs of $x + 3y^2 = 5$ and $x = -1$?

A. $\frac{20\sqrt{15}}{9}$ B. 16 C. $4\sqrt{2}$ D. $8\sqrt{2}$ E. NOTA

For questions 19-21, *Conic F (possibly degenerate) is described by the equation*

$$4x^2 - 10xy + 6y^2 + 14y + 2 = 0$$

19. Which of the following describes the graph of conic F?

A. Hyperbola B. Ellipse C. Two lines (Intersecting) D. Two lines (Parallel) E. NOTA

20. Let θ be the acute angle needed to rotate the coordinate axes counterclockwise by to eliminate the xy term of conic F. Find $\tan\theta$.

A. $\frac{1}{5}$ B. 5 C. $\frac{-\sqrt{26}-1}{5}$ D. $\frac{\sqrt{26}-1}{5}$ E. NOTA

21. Which of the following conics has the least eccentricity?

A. Conic F B. $36x^2 + 31y^2 = 1116$

C. $r = \frac{84}{11 - 11\sin\theta}$ D. $r = \frac{20}{22 + 7\cos\theta}$ E. NOTA

22. Let $\hat{\mathbf{u}}$ be the unit vector from $(-3,5)$ to $(7,-1)$. Find the dot product $\hat{\mathbf{u}} \cdot \langle 1,1 \rangle$.

A. $\frac{\sqrt{34}}{17}$ B. $-\frac{\sqrt{34}}{17}$ C. $\frac{1}{34}$ D. $-\frac{1}{34}$ E. NOTA

23. Rectangle $ABCD$ has side lengths $\overline{AB} = 143$ and $\overline{BC} = 26$. Points E and F are located on $\overline{CD}$, with $\overline{DE} > \overline{DF}$, such that $\overline{AE}$ and $\overline{AF}$ trisect $\angle CAD$. Find the length $\overline{BF}$.

A. $26\sqrt{26}$ B. $13\sqrt{53}$ C. $\frac{26\sqrt{130}}{3}$ D. $26\sqrt{29}$ E. NOTA

24. A graph is defined parametrically by $x(t) = \sqrt{t}$ and $y(t) = t - 1 + 16\sin(\frac{3}{2}\sqrt{t})$, for $t > 0$. On the x,y plane, how many x-intercepts does the graph have?

A. 1 B. 3 C. 4 D. 6 E. NOTA

25. Points $A(3,1,3)$, $B(4,1,6)$, and $C(-2,1,6)$ define a plane P. Point D is a distance of 2 from plane P. What is the volume of tetrahedron $ABCD$?

A. Not enough information B. 18 C. 12 D. 6 E. NOTA

26. The line $4x + 3y = 42$ divides the graph of $x^2 + y^2 - 12x - 22y = -121$ into two closed sections. What is the area of the larger section?

A. 24π B. $24\pi + 9\sqrt{3}$ C. $27\pi + 18$ D. $30\pi + 9\sqrt{3}$ E. NOTA

27. Line l has equation $l(t) = (1, 4, -3) + \langle 2, 4, -1 \rangle t$. At what value of t is the distance between line l and the point $(4, -2, 3)$ minimized?

A. $1/7$ B. 0 C. -1 D. $-8/7$ E. NOTA

28. Equilateral triangle ABC has vertex A at the point $(2, 2)$, vertex B at $(-1, -2)$, and vertex C in the second quadrant. The slope of the line connecting vertices B and C is in the form $\frac{p+q\sqrt{3}}{r}$, for integers p, q, and r, with $r > 0$ and relatively prime to at least one of $|p|$ and $|q|$. What is $p + q + r$?

A. 10 B. 16 C. -62 D. -34 E. NOTA

29. For $\vec{\mathbf{a}} = 3\mathbf{i} - 13\mathbf{j} - 8\mathbf{k}$, $\vec{\mathbf{b}} = -2\mathbf{i} - 5\mathbf{j}$, and $\vec{\mathbf{c}} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find the value of $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$.

A. -97 B. -65 C. 59 D. 67 E. NOTA

30. At how many points do the graphs of $r = 2022\theta$ and $r \sin \theta = 2022$ intersect for $|\theta| < \pi$?

A. Infinitely many B. 0 C. 2 D. 4 E. NOTA