Mu

Individual

Test #531

Directions:

1. Fill out the top section of the Individual Google Form answer sheet and select **Mu-Individual** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

On this test, answer choice E. should be chosen when None Of The Answers listed are correct.

As a general hint, sometime the questions prior to trickier questions are a clue as to how to solve them.

1. If $f(x) = \cos^4(x) - \sin^4(x)$, find $f'\left(\frac{\pi}{4}\right)$.

A. $-\sqrt{3}$ B. -1
C. $-\sqrt{2}$ D. -2 E. NOTA

2. Alex's favorite Statistics function is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, where $\sigma > 0$ is constant. Find the positive value of x for which $f(x)$ has a point of inflection.

A. 0 B. σ
C. $\sqrt{2}\sigma$ D. σ^2 E. NOTA

3. Evaluate:

$$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{3} + 5h\right) - \tan\left(\frac{\pi}{3} - 4h\right)}{3h} =$$

A. 4 B. 12
C. 3 D. $\frac{3}{4}$ E. NOTA

4. Given $f(x) = x^5 - 7x^3 + 3x^2 - x + 2022$, Which of the following pairs of behaviors best describes $f(x)$ at $x = 2$?

A. Increasing, Concave Down B. Decreasing, Concave Down
C. Increasing, Concave Up D. Decreasing, Concave Up E. NOTA

5. If $f(x) = x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8$, find $\frac{d}{dx}[f^{-1}(x)]$ evaluated at $x = 20$

A. 20 B. 26
C. $\frac{1}{26}$ D. $\frac{1}{20}$ E. NOTA

6. Find the largest value of $x > 0$ for which $\begin{bmatrix} x & 5x & 2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{bmatrix}$ does not have an inverse.

A. $\frac{2}{19}$

B. $\frac{17+\sqrt{137}}{38}$

C. 1

D. $\frac{17-\sqrt{137}}{38}$

E. NOTA

7. Find the minimal possible value of the determinant of $\begin{bmatrix} x & 5x & -2 \\ -3 & \frac{1}{x} & 2 \\ 1 & 3 & 1 \end{bmatrix}$ for $x > 0$. Note this is not the same matrix as the previous question.

A. $19 + 2\sqrt{38}$

B. $19 - 2\sqrt{38}$

C. $\frac{\sqrt{38}}{2}$

D. $\frac{\sqrt{38}}{19}$

E. NOTA

8. Evaluate

$$\int_{e^{e^e}}^{\infty} \frac{dx}{x(\ln(x))(\ln(\ln(x)))\,(\ln(\ln(\ln(x))))^{2022}}$$

A. $\frac{1}{2022}$

B. $-\frac{1}{2022}$

C. $\frac{1}{2021}$

D. $-\frac{1}{2021}$

E. NOTA

9. Find the maximum possible value of $\int_{-\pi}^{2\pi} f(x)\sin(x)\,dx$ if $|f(x)| \leq 2022$ everywhere.

A. 12132

B. 8088

C. 4044

D. 2022

E. NOTA

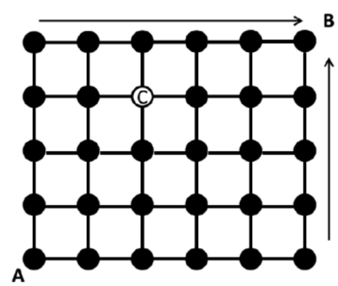
10. Evaluate

$$\lim_{n\to\infty}\left(\frac{n^2+2000n+4022}{n^2+2022n+4044}\right)^n$$

A. e^{-11} B. e^{-22}

C. e^{-44} D. e^{-88} E. NOTA

11. Consider the following grid of points:



Suppose that, starting at the point labeled A, Jae can go one step up or one step to the right at each move. This procedure is continued until Jae reaches the point labeled B. Given that Jae's path goes through the point C, what is the probability that Jae's first move was to the right, if every path is equally likely?

A. $\frac{1}{10}$ B. $\frac{1}{5}$

C. $\frac{2}{5}$ D. $\frac{3}{5}$ E. NOTA

12. Urn A has 6 red balls and 6 purple balls. Urn B has 8 red balls and 7 purple balls. Iris rolls a fair 6-sided die. If the outcome is one, she selects a ball from Urn A. Otherwise, she selects a ball from Urn B. Suppose Iris selected a purple ball. What is the probability that the die roll was one?

A. $\frac{1}{6}$ B. $\frac{1}{2}$

C. $\frac{7}{15}$ D. $\frac{3}{17}$ E. NOTA

For the next three questions, use the following information:

A **probability density function** is a piecewise continuous non-negative function $f(x)$ with the property that the area between this function and the x-axis is one. The probability of random variables attaining ranges of values can be determined using probability density functions, such that the probability of $a < X < b$ is given by $\int_a^b f(x)dx$. The **expectation**, or mean value, of a random variable with probability density $f(x)$ is given by $\int_{\mathbb{R}} xf(x)dx$.

13. Suppose $q(x) = \begin{cases} kx(2-x), 0 \le x \le 2 \\ 0, \text{Otherwise} \end{cases}$ is a probability density function. Find k.

A. $\frac{3}{8}$ B. $\frac{3}{4}$

C. $\frac{4}{3}$ D. $\frac{8}{3}$ E. NOTA

14. It is known that $r(x) = \begin{cases} \frac{1}{2}e^{-\sqrt{x}}, x \ge 0 \\ 0, \text{Otherwise} \end{cases}$ is a probability density function. Find the expectation of a random variable governed by this density function.

A. 6 B. 12

C. 24 D. 36 E. NOTA

15. There are three unlabeled computers in the computer lab. One of these computers (call it Computer A) has a boot-up time (in minutes) that is a random variable governed by the probability density $q(x)$ above. The remaining two computers have boot-up times (in minutes) that are random variables governed by the probability density $r(x)$ above. Saathvik picks a computer at random with equal probability and finds that the boot-up time was less than one minute. What is the probability that he is working on Computer A?

A. $\frac{e}{5e-8}$ B. $\frac{e}{9e-16}$

C. $\frac{1}{e}$ D. $\frac{1}{3}$ E. NOTA

16. If $Re\left(e^{i\alpha}\right) = \frac{5}{13}$ and $Im\left(e^{i\beta}\right) = \frac{3}{5}$, and both $-\frac{\pi}{2} < \alpha < 0$ and $\frac{\pi}{2} < \beta < \pi$ and then find $\tan(\alpha + \beta)$.

A. $-\frac{33}{56}$
B. $\frac{33}{56}$
C. $-\frac{63}{16}$
D. $\frac{63}{16}$
E. NOTA

17. Compute

$$\sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n!}$$

A. $\sqrt{e}\cos\left(\frac{\sqrt{3}}{2}\right)$
B. $\sqrt{e}\sin\left(\frac{\sqrt{3}}{2}\right)$
C. $\frac{\sqrt{e}}{2}$
D. $\frac{\pi\sqrt{e}}{3}$
E. NOTA

18. Consider the parabola $y = ax^2 + 2022x + 2022$. There exists exactly one circle which has its center on the x-axis and is tangent to the parabola at exactly two points. It turns out one of the tangent points is $(0, 2022)$. Determine the value of a.

A. $\frac{1}{2022}$
B. $-\frac{1}{2022}$
C. $\frac{1}{4044}$
D. $-\frac{1}{4044}$
E. NOTA

19. Define the functions $f_0, f_1, f_2, f_3, \dots, f_n, \dots$ to satisfy the relations $f_0 = \frac{1}{2}$ and $f_n' = f_{n-1}$. Furthermore, assume for all $n > 0$ that $f_n(0) = 0$. Evaluate

$$\sum_{n=0}^{\infty} f_n(\ln(2022))$$

A. 1011
B. 2022
C. 4044
D. 8088
E. NOTA

20. Let $f(x)$ be a continuous, differentiable function at $x = 0$ with $f(0) = 0$ and $f'(0) = 2022$. Evaluate $\lim_{x\to 0} \frac{f(x)}{x}$.

A. 1011 B. 2022

C. 4044 D. 8088 E. NOTA

21. Evaluate

$$\lim_{x\to 0} \frac{\sin^2(4044x)\tan^3(1011x)}{(\ln(2022\mathrm{x}+1))^5}$$

A. $\frac{1}{2}$ B. $\frac{1}{4}$

C. $\frac{1}{8}$ D. $\frac{1}{16}$ E. NOTA

22. Let $\mathcal{L}_1$ be the line in space with directional vector $< 1, -2, 2 >$ going through the point $(0, 1, 2)$. Let $\mathcal{L}_2$ be the line in space with directional vector $< 3, -1, -4 >$ going through the point $(3, -1, 2)$. Find the minimum distance between these lines.

A. 0 B. $\frac{2}{3}$

C. $\frac{10}{3}$ D. $\frac{2}{5}$ E. NOTA

23. Consider a line with direction vector $< 2, -2, 1 >$ going through the point $(2, 3, 4)$. Which of the following expressions represents the infinite cylinder in space that is the locus of all points a distance of 5 away from this line?

A. $(y + 2z - 11)^2 + (x - 2z + 6)^2 + (x + 2y - 10)^2 = 25$

B. $(x - 2)^2 + (y - 3)^2 + (z - 4)^2 = 25$

C. $(y + 2z - 11)^2 + (x - 2z + 6)^2 + (x + 2y - 10)^2 = 225$

D. $(y + 2z + 11)^2 + (x - 2z - 6)^2 + (x + 2y + 10)^2 = 25$

E. NOTA

24. Let $f(x)$ be a continuous, differentiable function satisfying $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)dx$.

A. $e-1$ B. e

C. e^{e-1} D. e^e E. NOTA

25. Let $f(x)$ be an odd, continuously differentiable function with $f(\pi) = 2022$.

Evaluate

$$\int_{-\pi}^{\pi} \frac{f'(x)}{1-\sin(x)+\sqrt{1+\sin^2(x)}}dx$$

A. 1011 B. 2022

C. 4044 D. 8088 E. NOTA

26. Find the area enclosed by the locus of all points that are exactly $\frac{1}{n-1}$ as far away from the point $(1,2)$ as they are from the line $y = -2x + 24$ for $n > 2$.

A. $\frac{40(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi$ B. $\frac{80(n-1)}{(n(n-2))^{\frac{3}{2}}}\pi$

C. $\frac{40\sqrt{n(n-2)}}{n-1}\pi$ D. $\frac{80\sqrt{n(n-2)}}{n-1}\pi$ E. NOTA

27. Find the rate of change of the area enclosed by the locus of all points that are exactly $\frac{1}{n-1}$ as far away from the point $(1,2)$ as they are from the line $y = -2x + 24$ for if n is increasing 48 units/hour and $n = 6$.

A. $-\frac{85\sqrt{24}\pi}{6}$ B. $-\frac{85\sqrt{24}\pi}{12}$

C. $\frac{85\sqrt{24}\pi}{6}$ D. $\frac{85\sqrt{24}\pi}{12}$ E. NOTA

28. If $t = \tan\left(\frac{\theta}{2}\right)$, and $0 < \theta < \frac{\pi}{2}$, then express $\sin(\theta) + \cos(\theta)$ in terms of t.

A. $\frac{1+t}{\sqrt{1+t}}$

B. $\frac{1+t}{\sqrt{1+t^2}}$

C. $\frac{(1+t)^2}{1+t^2}$

D. $\frac{1+2t-t^2}{1+t^2}$

E. NOTA

29. Evaluate:

$$\int_0^{\pi} \ln(1 - 4044\cos(x) + 2022^2)\, dx$$

A. $\pi \ln(4044)$

B. $2\pi \ln(4044)$

C. $\pi \ln(2022)$

D. $2\pi \ln(2022)$

E. NOTA

30. You made it to the end of the test! Congratulations! Now, evaluate

$$\lim_{x \to 4} \frac{\mathrm{x} + 4}{\mathrm{x}^2 + \sqrt{x - 4}}$$

A. $\frac{1}{2}$

B. 1

C. 2

D. 4

E. NOTA