Answers:

1. D	7. C	13. A	19. C	25. D
2. C	8. C	14. A	20. B	26. A
3. D	9. A	15. A	21. D	27. A
4. D	10. C	16. C	22. B	28. D
5. B	11. D	17. C	23. E ($\frac{2\pi}{3}$)	29. A
6. A	12. E (26)	18. C	24. C	30. D

Solutions:

1. D Distance = Arc length = $\int_{t_0}^{t_1}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt = \int_0^{6\sqrt{2}}\sqrt{(3t)^2+(t^2)^2}\,dt =$
$\int_0^{6\sqrt{2}}\sqrt{9t^2+t^4}dt = \int_0^{6\sqrt{2}} t\sqrt{9+t^2}\,dt$
$u = 9+t^2, du = 2t\,dt \rightarrow \int_9^{81}\frac{1}{2}\sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_9^{81} = \frac{1}{3}(729-27) = \boxed{234}$

2. C For any partitioning of 512 into two side lengths a and b, an angle of $\pi/2$ between them will maximize the area of the triangle. (This is seen by treating one side of the fence as the triangle's base; the altitude to this base is greatest when the angle between the sides of the fence is $\pi/2$.) Thus, the area is $\frac{1}{2}ab$.
In addition, from the AM-GM inequality, $\sqrt{ab} \le \frac{a+b}{2} \rightarrow \sqrt{ab} \le 256 \rightarrow$
$ab \le 2^{16} \rightarrow \frac{1}{2}ab \le 2^{15}$, so the maximum area is $\boxed{2^{15}}$. (The geometric mean and arithmetic mean of a and b are equal only in the case where $a = b$, so the area is maximized when $a = b = 256$.)
These conditions can also be discovered by reflecting the triangle over the line of the cliff to make a parallelogram, which has its maximum area when it is a square.

3. D $v(t) = \int a(t)dt = \langle 2t^2 + c_1, 3t^3 - 3t + c_2\rangle$
At $t = 0$, $v(0) = \langle c_1, c_2\rangle$; speed $= \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{{c_1}^2 + {c_2}^2} = 0 \rightarrow c_1 = c_2 = 0$
$v(1) = \langle 2, 0\rangle \rightarrow$ speed $= \sqrt{2^2+0^2} = \boxed{2}$

4. D The location of the figure on the coordinate plane does not affect its volume, so the center can be shifted to the origin to make the problem simpler.
$(x+6)^2 + 4(y-3)^2 = 16 \rightarrow x^2 + 4y^2 = 16 \rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$
This is an ellipse with semi-major axis $\sqrt{16} = 4$ and semi-minor axis $\sqrt{4} = 2$.

In general, the volume of a figure formed by equilateral triangle cross-sections perpendicular to the x-axis is $\frac{\sqrt{3}}{4}\int_{x_0}^{x_1}(B(x))^2dx$, where $B(x)$ is the base of each triangle.

$\frac{x^2}{16}+\frac{y^2}{4}=1 \rightarrow y^2=4-\frac{x^2}{4} \rightarrow y=\pm\frac{1}{2}\sqrt{16-x^2}$

The base of each triangle is the distance between the positive and negative values of y, so $B(x)=\frac{1}{2}\sqrt{16-x^2}-\left(-\frac{1}{2}\sqrt{16-x^2}\right)=\sqrt{16-x^2}$.

The endpoints of the horizontal axis of the ellipse are at $x=-4$ and $x=4$, so the volume is $\frac{\sqrt{3}}{4}\int_{-4}^{4}\left(\sqrt{16-x^2}\right)^2dx=(2)\frac{\sqrt{3}}{4}\int_0^4 16-x^2dx=\frac{\sqrt{3}}{2}\left(16x-\frac{x^3}{3}\right)\Big|_0^4$

$=\frac{\sqrt{3}}{2}\left(\left(16(4)-\frac{(4)^3}{3}\right)-(0)\right)=\frac{\sqrt{3}}{2}\left(\frac{128}{3}\right)=\boxed{\frac{64\sqrt{3}}{3}}$

5. B Let h represent half of the height of the cylinder, and let r represent the cylinder's radius. Since a right triangle connects the sphere's center, the center of the cylinder's base, and a point along the circumference of the cylinder's base, then

$$r^2+h^2=R^2.$$

$V_{cyl}=\pi r^2h=\pi(R^2-h^2)h=\pi R^2h-\pi h^3 \rightarrow$ maximum volume when

$\frac{dV_{cyl}}{dh}=\pi R^2-3\pi h^2=0 \rightarrow h\sqrt{3}=R$

h is half the height of the cylinder, so $R=\left(\frac{11\sqrt{3}}{2}\right)\sqrt{3}=\boxed{\frac{33}{2}}$.

6. A The amount A left of an original quantity P undergoing exponential decay with a rate r after a time t is given by the equation $A=Pe^{-rt}$. (This can be derived by solving the differential equation $\frac{dA}{dt}=-Ar$ with initial condition $A(0)=P$.)

After 20 minutes, $A=\frac{1}{2}P \rightarrow \frac{1}{2}P=Pe^{-20r} \rightarrow e^r=(\frac{1}{2})^{-1/20}=2^{1/20}$.

To find t when 1% of the original amount is left, plug in $\frac{1}{100}P$ for A:

$\frac{1}{100}P\ =Pe^{-rt} \rightarrow \frac{1}{100}=(e^r)^{-t} \rightarrow \frac{1}{100}=(2^{1/20})^{-t} \rightarrow 100=(2^{1/20})^t$

$\rightarrow 100^{20}=2^t \rightarrow t=\log_2(100^{20})=\log_2(10^{40})=\boxed{40\log_2 10}$

7. C The differential equation is separable by group factoring:

$\frac{dy}{dx}=\frac{5}{3xy+y+6x+2} \rightarrow \frac{dy}{dx}=\frac{5}{(y+2)(3x+1)} \rightarrow \int(y+2)\,dy=\int\frac{5}{3x+1}dx$

$\rightarrow \frac{y^2}{2}+2y=\frac{5}{3}\ln|3x+1|+C$

Plugging in the initial point $(1,2)$ to find C,

$\frac{(2)^2}{2}+2(2)=\frac{5}{3}\ln|3(1)+1|+C \rightarrow 6=\frac{5}{3}\ln 4+C \rightarrow C=6-\frac{10}{3}\ln 2$

Plug in $y=4$ and solve for x:

$\frac{(4)^2}{2} + 2(4) = \frac{5}{3}\ln|3x+1| + 6 - \frac{10}{3}\ln 2 \rightarrow 16 = \frac{5}{3}\ln|3x+1| + 6 - \frac{10}{3}\ln 2$
$\rightarrow 10 + \frac{10}{3}\ln 2 = \frac{5}{3}\ln|3x+1| \rightarrow \ln|3x+1| = 6 + 2\ln 2 \rightarrow 3x + 1 = e^{6+2\ln 2} =$
$4e^6 \rightarrow x = \boxed{\frac{4e^6-1}{3}}$

8. C Total distance $= \int_{t_0}^{t_1} v(t)\,dt = \int_2^{6\sqrt{3}} \frac{1}{\left(\frac{t}{16}\right)^{\frac{2}{3}}\sqrt{1-\left(\frac{t}{16}\right)^{\frac{2}{3}}}}\,dt$

$u = \frac{t}{16}, du = \frac{1}{16}dt \rightarrow 16\int_{\frac{1}{8}}^{\frac{3\sqrt{3}}{8}} \frac{1}{u^{\frac{2}{3}}\sqrt{1-u^{\frac{2}{3}}}}\,du = 16\int_{\frac{1}{8}}^{\frac{3\sqrt{3}}{8}} \frac{1}{u^{\frac{2}{3}}\sqrt{1-(u^{\frac{1}{3}})^2}}\,du$

$v = u^{\frac{1}{3}}, dv = \frac{1}{3}u^{-\frac{2}{3}}\,du \rightarrow 48\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-v^2}}\,dv = 48\sin^{-1} v\,\Big|_{1/2}^{\sqrt{3}/2} = 48\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \boxed{8\pi}$

9. A Let x be the amount of 20% butter sauce used and y be then amount of 50% butter sauce used. Then $x + y = 100$, and $0.2x + 0.5y = 0.41(100) = 41$
$\rightarrow 0.4x + y = 82$.
By elimination, $0.6x = 18 \rightarrow x = 30 \rightarrow y = 100 - (30) = 70$
$|70 - 30| = \boxed{40}$

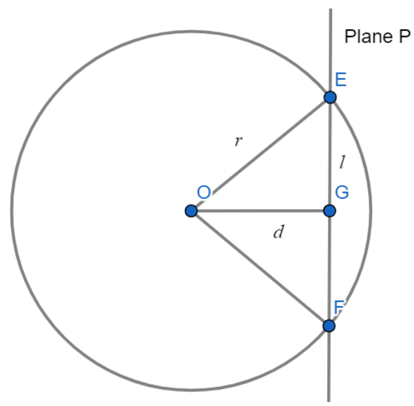
10. C The Mean Value Theorem states that for a function $f(x)$ continuous on an interval $[a, b]$ and differentiable on (a, b), there must be at least one value c for $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Thus, for some $x = c$ on the interval $(2, 5)$, $f'(x)$ must equal $\frac{f(5)-f(2)}{5-2} = \frac{20\pi-14\pi}{3} = \frac{6\pi}{3} = \boxed{2\pi}$

11. D $h(t) = 70 + 50\sin(\frac{\pi}{9}t + \frac{3\pi}{2}) = 45 \rightarrow 50\sin(\frac{\pi}{9}t + \frac{3\pi}{2}) = -25 \rightarrow$
$\sin(\frac{\pi}{9}t + \frac{3\pi}{2}) = -\frac{1}{2} \rightarrow \frac{\pi}{9}t + \frac{3\pi}{2} = \frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k, k \in \mathbb{Z}$
The smallest value of $\frac{\pi}{9}t + \frac{3\pi}{2}$ for which $\sin(\frac{\pi}{9}t + \frac{3\pi}{2}) = -\frac{1}{2}$ and $t > 0$ is $\frac{11\pi}{6}$, so at the second time t_2 that $h(t) = 45$, $\frac{\pi}{9}t + \frac{3\pi}{2}$ is $\frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$ (the next smallest value).
The instantaneous rate of change of Sharay's height is $h'(t) = 50\left(\frac{\pi}{9}\right)\cos\left(\frac{\pi}{9}t + \frac{3\pi}{2}\right)$
$\rightarrow h'(t_2) = \frac{50\pi}{9}\cos\left(\frac{19\pi}{6}\right) = \frac{50\pi}{9}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{-25\pi\sqrt{3}}{9}}$

12. E Let x be the x-coordinate of the circle's greater intersection with the x-axis. The circle's center $(1, 5)$, the point $(1, 0)$, and the point $(x, 0)$ form a right triangle. The vertical leg of this triangle is 5, the horizontal leg is $x - 1$, and the hypotenuse is the

circle's radius. Then by the Pythagorean theorem, $(x-1)^2+5^2=r^2 \rightarrow 2(x-1)\frac{dx}{dt}=2r\frac{dr}{dt} \rightarrow 2(12)\frac{dx}{dt}=2(13)(24) \rightarrow \frac{dx}{dt}=\boxed{26}$

13. A This problem is analogous to question 12.
Let O be the origin and P be the plane $2x+y-2z=9$.
Consider a cross-section of the sphere taken through the sphere's center O at right angles to the plane P, as pictured below. Once the sphere grows beyond plane P, a chord $\overline{EF}$ crosses the circular cross-section, a distance of d from the sphere's center. The radius $\overline{OE}$, the distance d, and half of the chord $\overline{EF}$ form a right triangle, with hypotenuse $\overline{OE}$. By the Pythagorean theorem, $d^2+l^2=r^2$, where r is the radius and l is half the length of $\overline{EF}$.



Using the point-to-plane distance formula, the distance d from O to plane P is $\frac{|Aa+Bb+Cc-D|}{\sqrt{A^2+B^2+C^2}}=\frac{|2(0)+1(0)-2(0)-9|}{\sqrt{2^2+1^2+(-2)^2}}=\frac{|-9|}{\sqrt{9}}=3$. Thus, as the sphere grows, r and l obey the relationship $3^2+l^2=r^2 \rightarrow 9+l^2=r^2$. When $r=5$, $l^2=5^2-9=16 \rightarrow l=4$.
The volume V of the sphere increases at a constant rate of 8 $units^3/s$. Thus, when $r=5$, $V=\frac{4\pi}{3}r^3 \rightarrow \frac{dV}{dt}=4\pi r^2\frac{dr}{dt} \rightarrow 8=4\pi(25)\frac{dr}{dt} \rightarrow \frac{dr}{dt}=\frac{2}{25\pi}$.
Since $9+l^2=r^2$, then $2l\frac{dl}{dt}=2r\frac{dr}{dt} \rightarrow 2(4)\frac{dl}{dt}=2(5)(\frac{2}{25\pi}) \rightarrow \frac{dl}{dt}=\frac{1}{10\pi}$.
Finally, the cross-section formed by the sphere's intersection with plane P is simply a circle with radius l, so $A=\pi l^2 \rightarrow \frac{dA}{dl}=2\pi l\frac{dl}{dt}=2\pi(4)\left(\frac{1}{10\pi}\right)=\boxed{4/5}$.

14. A By noticing Pascal's triangle in the coefficients, the problem becomes much simpler.
$x^3-3x^2+3x-2023=0 \rightarrow (x-1)^3=2022$
There are three roots to this equation, two of which are clearly imaginary. If the sum of all roots is known, the sum of the nonreal roots can be found by subtracting the one real root from the total sum. From Vieta's formulas, the sum of all roots is $-\frac{-3}{1}=3$, and the real root is quickly found to be $1+\sqrt[3]{2022}$.

$3-\left(1+\sqrt[3]{2022}\right)=2-\sqrt[3]{2022}$
$12^3<2022<13^3 \rightarrow 12<\sqrt[3]{2022}<13 \rightarrow -11<2-\sqrt[3]{2022}<-10$
$\rightarrow \lfloor 2-\sqrt[3]{2022}\rfloor = \boxed{-11}$

15. A $L(n)=\lim\limits_{p\to 0} A(n)=\lim\limits_{p\to 0}\frac{n+p+\frac{1}{n+p}-\left(n+\frac{1}{n}\right)}{p}=\lim\limits_{p\to 0}\frac{f(n+p)-f(n)}{p}$, where $f(n)=n+\frac{1}{n}$.
This is recognizable as the limit definition of the derivative of $f(n)$.
$L(n)=f'(n)=1-\frac{1}{n^2}\rightarrow L(21)=f'(21)=1-\frac{1}{(21)^2}=1-\frac{1}{441}=\boxed{\frac{440}{441}}$

16. C Since we want the total volume and $\sin x$ changes sign at $x=\pi$, the integral should be computed as two separate parts: from $x=0$ to $x=\pi$ and from $x=\pi$ to $x=2\pi$. By the washer method, the total volume is:
$V=\pi\int_0^{\pi}\left(x\sqrt{|\sin x|}\right)^2dx+\pi\int_{\pi}^{2\pi}\left(x\sqrt{|\sin x|}\right)^2dx=\pi\int_0^{\pi}x^2\sin x\,dx-\pi\int_{\pi}^{2\pi}x^2\sin x\,dx$ ($\sin x$ is negative on the second interval, so the integral must be negated to remove the absolute value bars.)
By parts, $\int x^2\sin x\,dx=-x^2\cos x+2x\sin x+2\cos x+C \rightarrow$
$V=\pi\left(-x^2\cos x+2x\sin x+2\cos x\,\Big|_0^{\pi}\right)-\pi\left(-x^2\cos x+2x\sin x+2\cos x\,\Big|_{\pi}^{2\pi}\right)=\pi\left((\pi^2-2)-2\right)-\pi\left((-4\pi^2+2)-(\pi^2-2)\right)=\pi^3-4\pi+5\pi^3-4\pi=\boxed{6\pi^3-8\pi}$

17. C The cosine of the acute angle θ between two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is given by

$$\cos\theta=\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{|\vec{\mathbf{u}}||\vec{\mathbf{v}}|}$$

Then the cosine of the acute angle between the vectors $\langle 3,2,6\rangle$ and $\langle 4,0,3\rangle$ is
$\frac{(3)(4)+(2)(0)+(6)(3)}{\sqrt{3^2+2^2+6^2}\sqrt{4^2+0^2+3^2}}=\frac{12+0+18}{\sqrt{49}\sqrt{25}}=\frac{30}{(7)(5)}=\boxed{\frac{6}{7}}$

18. C $g(t)=2\cos t\,(\sec t+\sin t)=2+\sin 2t$
$\mathcal{L}[g](s)=\int_0^{\infty}e^{-st}g(t)dt=\int_0^{\infty}e^{-st}(2+\sin 2t)dt$
Break the integral into two separate integrals:

$$I+J=\int_0^{\infty}2e^{-st}dt+\int_0^{\infty}e^{-st}\sin 2t\,dt$$

$I=\int_0^{\infty}2e^{-st}dt=-\frac{2e^{-st}}{s}\Big|_0^{\infty}=\left(\lim\limits_{t\to\infty}-\frac{2e^{-st}}{s}\right)-\left(-\frac{2(1)}{s}\right)=0+\frac{2}{s}=\frac{2}{s}$
By parts:
$J=\int_0^{\infty}e^{-st}\sin 2t\,dt=\left(-\frac{e^{-st}\sin 2t}{s}-\frac{2e^{-st}\cos 2t}{s^2}\Big|_0^{\infty}\right)-\frac{4}{s^2}J$
$\left(1+\frac{4}{s^2}\right)J=\left(\lim\limits_{t\to\infty}-\frac{e^{-st}\sin 2t}{s}-\frac{2e^{-st}\cos 2t}{s^2}\right)-\left(-\frac{(1)(0)}{s}-\frac{2(1)(1)}{s^2}\right)=0+\frac{2}{s^2}=\frac{2}{s^2}$

$J = \frac{\frac{2}{s^2}}{1+\frac{4}{s^2}} = \frac{2}{4+s^2}$

$G(s) = I + J = \frac{2}{s} + \frac{2}{4+s^2} \rightarrow G(2) = \frac{2}{2} + \frac{2}{4+2^2} = 1 + \frac{1}{4} = \boxed{5/4}$

19. C The nth smallest positive multiple of a positive integer a is simply $a \cdot n$, so $M(a,n) = an$.

$$S = \sum_{k=1}^{13}\sum_{j=1}^{200} M(k,j) = \sum_{k=1}^{13}\sum_{j=1}^{200} kj$$

$$= \big((1+2+3+\cdots+200) + (2+4+6+\cdots+400) + \cdots$$
$$+ (13+26+39+\cdots+13\cdot 200)\big)$$
$$= 1(1+2+3+\cdots+200) + 2(1+2+3+\cdots+200) + \cdots$$
$$+ 13(1+2+3+\cdots+200)$$
$$= (1+2+3+..+13)(1+2+3+\cdots+200)$$
$$= \left(\frac{13(14)}{2}\right)\left(\frac{200(201)}{2}\right) = 91(20100) = 1{,}829{,}100$$

The sum of the digits of 1,829,100 is $\boxed{21}$.

20. B <u>Solution 1</u>

Arbitrarily, let the bottom of the rope (where Milaan's window is) be a height of 100 m above the ground. When the rope swings down, a point on the rope at a height of $100 + x$ m above the ground moves to a new point $100 - x$ m above the ground. Then a point x m from the bottom of the rope moves a total distance of $(100 + x) - (100 - x) = 2x$ m.

$W = F \cdot d = ma \cdot d$, where d is the distance moved and m is the rope's mass.

The acceleration a in the scenario is the acceleration due to gravity, which is $10\ m/s^2$. In addition, the linear density λ is 1 kg/m $\rightarrow \frac{m}{L} = \lambda = 1 \rightarrow m = L$, where L is the length of the rope. (Treat L as a variable.)

Then $W = (L)(10) \cdot d \rightarrow dW = 10d\ dL$. Integrating along the 100 m length of the rope, we can substitute $2x$ for the distance d moved by an infinitesimal length dL of the rope:

$$W = \int_0^{100} 10(2x)\ dL$$

This is the same as integrating with respect to x, since both L and x measure the distance from the bottom of the rope.

$$W = \int_0^{100} 10(2x)\ dx = 10x^2 \Big|_0^{100} = 10(100)^2 = \boxed{100{,}000}$$

Solution 2

$W = F \cdot d = ma \cdot d = 10m \cdot d$

This scenario is equivalent to the scenario where the 100 kg mass of the rope is concentrated entirely at the rope's center of mass. The center of mass moves a distance of 100 m, so therefore the total work done is simply $10(100)(100) = \boxed{100{,}000}\ kg \cdot m^2/s^2$.

21. D Separate the problem into the work done on each of the rope and the bucket.

Rope:

As the rope is pulled up 100 m to Milaan's window, a point x m from the bottom of the rope is pulled distance of $100 - x$ against gravity. By the same reasoning as in question 20, the total work done is

$$W = \int_0^{100} 10(100 - x)\ dx = 10\left(100x - \frac{x^2}{2}\right)\Big|_0^{100} = 10\left(100^2 - \frac{100^2}{2}\right)$$

$$= 10\left(\frac{100^2}{2}\right) = 5(10{,}000) = 50{,}000\ \ kg \cdot m^2/s^2$$

Or, equivalently, the center of mass is moved 50 m, so $W = 10(100)(50) = 50{,}000\ \ kg \cdot m^2/s^2$.

Bucket:

The bucket's mass m in kg at time t after it begins being lifted is given by $m(t) = 5 + (35 - 2t) = 40 - 2t$. It takes 12.5 seconds to lift the bucket the full 100 m, so the constant rate at which the bucket's height h above the ground changes is $\frac{100}{12.5} = 8$ m/s → $h = 8t$. Therefore, m at time t is also equal to $40 - 2\left(\frac{h}{8}\right) = 40 - \frac{h}{4}$, and the force F is $ma = (40 - \frac{h}{4})(10)$. For a variable force, the work $W = \int F\,dx$, where x is distance, so

$$W = \int_0^{100} 10(40 - \frac{h}{4})\,dh = 10\left(40h - \frac{h^2}{8}\right)\Big|_0^{100} = 10\left(40(100) - \frac{100^2}{8}\right)$$

$$= 10(4{,}000 - 1{,}250) = 10(2{,}750) = 27{,}500\ \ kg \cdot m^2/s^2$$

This is also the answer obtained by calculating the average mass of the bucket during its constant ascent, 27.5 kg, and multiplying by $a = 10\ m/s^2$ and $d = 100$ m.

Thus, the total work done by Milaan on both the rope and the bucket is $50{,}000 + 27{,}500 = \boxed{77{,}500}\ kg \cdot m^2/s^2$.

22. B The rate of change of the volume V is proportional to the surface area $A = 4\pi r^2$, so $\frac{dV}{dt} = -kA = -4\pi k r^2$, for some constant $k > 0$.
In addition, $V = \frac{4\pi}{3}r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.
Setting these equations for $\frac{dV}{dt}$ equal,
$4\pi r^2 \frac{dr}{dt} = -4\pi k r^2 \rightarrow \frac{dr}{dt} = -k \rightarrow r(t) = -kt + C$
$r(0) = 8 = -k(0) + C \rightarrow C = 8$
$r(4) = 6 = -k(4) + 8 \rightarrow k = \frac{1}{2}$
$r(8) = -\frac{1}{2}(8) + 8 = -4 + 8 = \boxed{4}$

23. E The segment of the line $x + 2y = 2$ in the first quadrant is the line segment with endpoints $(0, 1)$ and $(2, 0)$. Revolving this segment around the x-axis forms a cone-shaped surface with a radius of 1 and a height of 2. The volume V of a cone is given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}(1)^2(2) = \boxed{\frac{2\pi}{3}}$

24. C The distance d between the Kyles is simply $f(x) - (-3) = f(x) + 3$.
$\frac{dd}{dt} = \frac{d}{dt}(f(x) + 3) = \frac{d}{dt}f(x) = \frac{df}{dx} \cdot \frac{dx}{dt}$. $\frac{dx}{dt} = 1$ since both Kyles' x-coordinates increasing at a rate of 1 unit/s, so $\frac{df}{dx} \cdot \frac{dx}{dt} = \frac{df}{dx} = f'(x) \rightarrow \frac{dd}{dt} = f'(x)$.
$f(x) = -\frac{1}{40}x^4 + \frac{3}{20}x^3 + \frac{3}{5}x^2 + \frac{7}{5}x + \frac{17}{4}$
$f'(x) = -\frac{1}{10}x^3 + \frac{9}{20}x^2 + \frac{6}{5}x + \frac{7}{5}$
$f''(x) = -\frac{3}{10}x^2 + \frac{9}{10}x + \frac{6}{5}$
To find the inflection point of $f(x)$ where $x > 0$,
$f''(x) = -\frac{3}{10}x^2 + \frac{9}{10}x + \frac{6}{5} = 0 \rightarrow x^2 - 3x - 4 = 0 \rightarrow (x - 4)(x + 1) = 0 \rightarrow$
$x = 4, -1 \rightarrow x = 4$
Then $\frac{dd}{dt}$ at this point is $f'(4) = -\frac{1}{10}(4)^3 + \frac{9}{20}(4)^2 + \frac{6}{5}(4) + \frac{7}{5} = -\frac{32}{5} + \frac{36}{5} + \frac{24}{5} + \frac{7}{5} = \frac{35}{5} = \boxed{7}$

25. D The volume V at time t is given by $V(t) = 22 + \int_1^t r(x)\,dx$. ($x$ is used here to avoid having t twice.)
$V(2) = 22 + \int_1^2 r(x)\,dx = 22 + \int_1^2 \frac{3x^7 + 14x^6 + 3x + 2}{x^7 + x}\,dx$
$\int_1^2 \frac{3x^7 + 14x^6 + 3x + 2}{x^7 + x}\,dx = \int_1^2 \frac{3(x^7 + x) + 2(7x^6 + 1)}{x^7 + x}\,dx = \int_1^2 3 + \frac{2(7x^6 + 1)}{x^7 + x}\,dx = 3(2 - 1) + \int_1^2 \frac{2(7x^6 + 1)}{x^7 + x}\,dx = 3 + \int_1^2 \frac{2(7x^6 + 1)}{x^7 + x}\,dx$

$u = x^7 + x,\ du = 7x^6 + 1\ dx \rightarrow \int_2^{130} \frac{2}{u} du = 2\ln u \Big|_2^{130} = 2(\ln 130 - \ln 2) = 2\ln 65$
$V(2) = 22 + 3 + 2\ln 65 = 25 + 2\ln 65 \rightarrow 25 + 2 + 65 = \boxed{92}$

26. A The AM-GM inequality states that the arithmetic mean of a set of nonnegative numbers is greater than or equal to the geometric mean of the set. Let the 3 nonnegative roots of $a(x)$ be r, s, and t. Then by AM-GM, $\frac{r+s+t}{3} \geq \sqrt[3]{rst}$. From Vieta's formulas, $r + s + t = -\frac{b}{a} = -\frac{(-6)}{1} = 6$, so $\sqrt[3]{rst} \leq \frac{6}{3} = 2 \rightarrow rst \leq 8$. From Vieta's formulas again, $rst = -\frac{d}{a} = -d \rightarrow d = -rst$, so $rst \leq 8 \rightarrow -rst \geq -8 \rightarrow d \geq \boxed{-8}$

27. A The probability of a player winning on his turn is the probability of drawing a black ball multiplied by the probability that the other player drew a white ball on the previous turn. Let the turn number be n. Then we have the following probabilities for winning on each turn:

Turn $= n$	Zach	Lou
1	$\frac{1}{2}$	
2		$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{3}{2\cdot 4}$
3	$\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{5}{6}\right) = \frac{5}{2\cdot 4\cdot 6}$	
4		$\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)\left(\frac{7}{8}\right) = \frac{7}{2\cdot 4\cdot 6\cdot 8}$
5	$\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)\left(\frac{1}{8}\right)\left(\frac{9}{10}\right) = \frac{9}{2\cdot 4\cdot 6\cdot 8\cdot 10}$	

From the chart, it is clear that the probability of the player taking his turn on turn n winning on turn n is $\frac{2n-1}{2^n \cdot n!} = \frac{2n}{2^n \cdot n!} - \frac{1}{2^n \cdot n!} = \frac{1}{2^{n-1}\cdot(n-1)!} - \frac{1}{2^n \cdot n!}$. Since Zach's turns are on odd values of n, the infinite sum of $\frac{1}{2^{n-1}\cdot(n-1)!} - \frac{1}{2^n \cdot n!}$ for only odd values of n is the total probability that Zach will win.

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}\cdot(n-1)!} - \frac{1}{2^n \cdot n!}$ for odd $n = \left(\frac{1}{2^0\cdot 0!} - \frac{1}{2^1\cdot 1!}\right) + \left(\frac{1}{2^2\cdot 2!} - \frac{1}{2^3\cdot 3!}\right) + \cdots$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!} = \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} = e^{-1/2} = \boxed{\sqrt{e}/_e}$$

28. D Regardless of the initial population, the rate of population growth for a logistic growth curve $\frac{dP}{dt} = kP(1 - \frac{P}{N})$ is maximized at the inflection point of the curve, which is where $P = N/2$. $N = 400$ in the given equation, so the population is growing fastest when $P = \frac{400}{2} = \boxed{200}$

29. A Note that there are 8 Es and 2 Bs in the 13-letter word. Then there are $\frac{13!}{8!2!} = 77{,}220$ total unique permutations. Find the number that do not begin with E by first finding the complement (the arrangements which do begin with E) and then subtracting these from the total. After placing an E in the first spot, there are $\frac{12!}{7!2!} = 47{,}520$ permutations for the remaining 12 letters in the remaining 12 spots, so there are $47{,}520$ arrangements which do begin with E. The number not beginning with E is thus $77{,}220 - 47{,}520 = \boxed{29{,}700}$.

30. D Using $z = r \cdot (\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$,

$2|z|^7 = z^6 + (\,\overline{z}\,)^6 \rightarrow 2|r\operatorname{cis}\theta|^7 = (r\operatorname{cis}\theta)^6 + (\,\overline{r\operatorname{cis}\theta}\,)^6$

$|r\operatorname{cis}\theta| = r$

$\overline{r\operatorname{cis}\theta} = r \cdot (\cos\theta - i\sin\theta) = r \cdot (\cos(-\theta) + i\sin(-\theta)) = r\operatorname{cis}(-\theta)$

$$\begin{aligned} 2|r\operatorname{cis}\theta|^7 = 2r^7 &= (r\operatorname{cis}\theta)^6 + (\,r\operatorname{cis}(-\theta)\,)^6 \\ &= (r^6\cos(6\theta) + r^6 i\sin(6\theta)) + (r^6\cos(-6\theta) + r^6 i\sin(-6\theta)) \\ &= r^6\cos(6\theta) + r^6 i\sin(6\theta) + r^6\cos(6\theta) - r^6 i\sin(6\theta) \\ &= 2r^6\cos(6\theta) \end{aligned}$$

$2r^7 = 2r^6\cos(6\theta) \rightarrow r = \cos(6\theta)$

This is a rose curve in the form $r = a\cos(b\theta)$ where b is even, so it has $2(6) = 12$ petals. In addition, the maximum distance of a petal is $a = 1$. Since Sammy starts at the polar point $(r, \theta) = (1, 0)$, she begins at the endpoint of a petal. As she skates along the curve, she will pass through the origin every time she reaches the endpoint of a new petal. Therefore, when she returns to her starting point $(1, 0)$, she will have passed through the origin 1 time for every petal, for a total of $\boxed{12}$ times.