Alpha

Applications

Test #421

Directions:

1. Fill out the top section of the Round 2 Google Form answer sheet and select **Alpha-Applications** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select "E) NOTA" and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like "which of the following could be" or "what is one solution of", an answer choice providing one of the possibilities is considered to be correct. Do not select "E) NOTA" in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select "E) NOTA" in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

The answer choice E. NOTA indicates that 'none of these answers' are correct. Problems are not necessarily in order of increasing difficulty, so don't be afraid to skip around. Good luck and have fun!

1. Legosi really likes egg sandwiches, and he eats them three days a week. The sandwiches on Fridays are the best, so he always gets one on Friday. The sandwiches on Wednesdays are the worst, so he never gets one on Wednesday. How many possible combinations of three days are there for him to eat his egg sandwiches?
   A. 10 B. 20 C. 15 D. 35 E. NOTA

2. Sound is composed of waves which have a velocity v that is only dependent on the medium they propagate through. The frequency f and wavelength λ of a sound wave satisfy $v = \lambda f$. Miguno is playing his guitar for Durham, and he plays a perfect fifth consisting of the notes A ($f = 440$ Hz) and E ($f = 660$ Hz). What is the ratio of the wavelength of A to the wavelength of E? Assume the waves are both propagating through air.
   A. $2/3$ B. 1 C. $9/4$ D. $3/2$ E. NOTA

3. Hina is floating in the clouds when Hodaka suddenly barrels past, falling at a constant speed of 15 m/s. After two seconds of processing what is going on, Hina dives after him at a constant speed of 20 m/s. How many seconds does it take Hina to catch up to Hodaka once she starts moving?
   A. 1.5 B. 6 C. 8 D. 10 E. NOTA

4. The amount of work W done by a force $\vec{F}$ over a displacement $\vec{x}$ is given by $W = \vec{F} \cdot \vec{x}$. Captain Dietfried and Major Gilbert are pushing a box of supplies. Dietfried is applying a force of magnitude F in the $+x$ direction, while Gilbert is applying a force of magnitude $2F$ in the $+y$ direction. The box moves in the direction of their net force. What is the ratio of the amount of work Dietfried does to the amount of work Gilbert does?
   A. $4:1$ B. $1:4$ C. $2:1$ D. $1:2$ E. NOTA

5. Mitsuha is riding her bike to the top of Itomori Mountain to meet Taki. Her bike has wheels of radius $1/\pi$ meters, and she must travel 1 kilometer to get to Taki. If her bike wheels complete two rotations per second and roll without slipping, how many minutes does it take Mitsuha to reach Taki, rounded to the nearest integer?
   A. 4 B. 2 C. 8 D. 6 E. NOTA

6. Haku is attempting to sneak across a bridge. To avoid being heard, he takes 4 steps in 4 seconds, stands still for 3 seconds, takes another 4 steps in 4 seconds, stands still for another 3 seconds, and so on continuing this alternating pattern. Given that he can cross the bridge in 100 steps, how many seconds does it take him to cross the bridge?
   A. 100 B. 172 C. 175 D. 178 E. NOTA

7. Retsuko is just trying to get her work done, but Ton keeps giving her more and more to do! On day 1, she must complete 11 tasks. On day 2, this number increases to 13. Days 3 and 4 bring 15 and 17 tasks, respectively, and this pattern of increasing by 2 every day continues. On which day does Retsuko complete her 2022nd task?
   A. 46 B. 40 C. 45 D. 41 E. NOTA

8. Luffy heads out on his ship at a speed of 2 knots (nautical miles per hour) and a bearing 15° south of east. At the same time, Zoro heads out from the same location at another ship at a speed of 3 knots and a bearing 15° west of north. What is the square of the distance between Luffy and Zoro after 2 hours, in nautical miles squared?
   A. 76 B. 28 C. $52 + 24\sqrt{3}$ D. $52 - 24\sqrt{3}$ E. NOTA

9. Don throws a rock off a cliff. The rock has a path modeled by the parametric equations $x(t) = 60t$ and $y(t) = -32t^2 + 48t + 5$, where t is the number of seconds after the rock leaves Don's hand, $(0,0)$ is the edge of the cliff Don is standing on, and all position measurements are in feet. If the rock lands at the edge of an opposing cliff 139 feet below Don's feet (at $y = -139$), what is the horizontal distance between the two cliffs, in feet?
   A. 3 B. 90 C. 180 D. 120 E. NOTA

10. Voss's height (in inches) fluctuates as a sinusoidal function of the chapter number with amplitude A and period T. Given that Voss's height is at its maximum value of 27 inches in chapter 1 and at its minimum value of 15 inches in chapter 123, find the sum of A and all possible integer values of T.
   A. 130 B. 145 C. 254 D. 269 E. NOTA

11. Shirou is falling from the top of the Sylvasta Pharmaceuticals building, which is 500 meters tall. His distance from the ground t seconds after falling off can be modeled by the equation $y(t) = 500 - 5t^2$. How fast is he travelling when he hits the ground at $y = 0$, in meters per second? (Hint: velocity increases linearly due to gravity as he falls. And don't worry, Shirou will be okay—his plot armor is ridiculous)
   A. 20 B. 50 C. 100 D. 10 E. NOTA

12. During a game of tag, Thoma and Lannion reach a fork in their path and decide to split up to avoid both being caught. Thoma turns counterclockwise by an acute angle α satisfying $\cos(\alpha) = 4/5$, while Lannion turns clockwise by an acute angle β satisfying $\sin(\beta) = 5/13$. What is the cosine of the acute angle between their new paths?
   A. $56/65$ B. $33/65$ C. $63/65$ D. $16/65$ E. NOTA

13. The comet Tiamat orbits the Earth along an elliptical path with the Earth at one of its foci. The distance between Tiamat and the Earth at its perigee (the point in its orbit closest to the Earth) is 1% of the distance between Tiamat and the Earth at its apogee (the point in its orbit furthest from the Earth). What is the eccentricity of Tiamat's orbit? (The *eccentricity* of an ellipse is the ratio of the distance between its foci to the distance between its vertices)

A. $99/100$ B. $49/50$ C. $100/101$ D. $99/101$ E. NOTA

14. Nazuna, Pinga, and Meteor are flying in the sky above Anima City while Mayor Rose looks at them from the ground. At one instant, Rose is located at the point $R(2,0,1)$, Pinga is located at $P(9,2,6)$, and Meteor is located at $M(6,2,1)$. If Nazuna is located at point N such that $m\angle NRP = m\angle NRM = 90°$, which of the following points could be Nazuna's location?

A. $(-8,20,7)$ B. $(-10,20,6)$ C. $(-10,27,6)$ D. $(-12,20,5)$ E. NOTA

15. Phil is taking his daily Grace Field math test and has reached the last question, where he must identify the graph of $r = 8 + 5\sin(\theta) + 6\cos(\theta)$. He knows it's a limaçon, but which type?

A. Cardioid B. Convex C. Inner loop D. Dimpled E. NOTA

16. Chihiro is trying to figure out how many susuwatari there are in the boiler room. She tells them to line up in rows of 6, but there end up being 5 left over. She then tells them to line up in rows of 7, but there end up being 6 left over. Finally, she tells them to line up in rows of 8, but there end up being 7 left over. What is the sum of the digits of the smallest possible number of susuwatari in the boiler room?

A. 15 B. 13 C. 11 D. 9 E. NOTA

17. Yahya grows carrots in his penthouse. His soil has a special fertilizer that makes his carrots grow in an interesting way. At the end of each day, each mature carrot produces a new baby carrot, which will grow into a mature carrot after one day. Suppose Yahya plants a single mature carrot at the start of day 0. How many mature carrots will Yahya have on day 30? For example, on day 1, Yahya will have one mature carrot and one baby carrot, and on day 2, Yahya will have two mature carrots and one baby carrot. (Let F_n denote the Fibonacci sequence, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$)

A. F_{28} B. F_{29} C. F_{30} D. F_{31} E. NOTA

18. Pina thought it would be funny to tease Bill with a laser pointer (no cat is immune, no matter how big). Bill is initially at a vertex of a regular hexagon of side length 1. Pina shines the laser point at one of the other five vertices of the hexagon chosen at random. Bill then runs in a straight line to the point. The instant Bill "catches" the point, Pina randomly selects a new vertex and shines the point there (he will never choose the vertex Bill is currently on). Given that Pina shines the laser 50 times before Bill catches on to what's happening, what is the expected total distance Bill will run?

A. $20 + 10\sqrt{3}$ B. $20 + 20\sqrt{3}$ C. $40 + 20\sqrt{3}$ D. $40 + 40\sqrt{3}$ E. NOTA

19. Ibuki is driving directly towards the Black Market Tunnel, the entrance of which can be modeled by the equation $x^2 + 8y = 64$. The largest cross section of Ibuki's car taken parallel to the entrance of the tunnel can be approximated as a square of side length 2, the base of which coincides with the ground ($y = 0$). Ignoring traffic laws (this is the Black Market, after all), the set of all x-coordinates of the center of mass of the car that allow Ibuki to safely pass through the entrance of the tunnel form an interval of length L. Find L.

A. $8\sqrt{3} - 2$ B. $4\sqrt{13} - 2$ C. $4\sqrt{3} - 1$ D. $2\sqrt{13} - 1$ E. NOTA

20. Michiru is NIGHT RUNNING through Anima City, which is arranged in a perfectly square grid of blocks. Michiru runs in a straight line from (0,0) to (3,1), then turns an angle θ counterclockwise and runs in a straight line from (3,1) to (8,3). If she then turns the same angle θ counterclockwise and continues running in a straight line, what will her x-coordinate be the next time she hits a point with integer coordinates?

A. 83 B. 91 C. 221 D. 47 E. NOTA

21. The strength of Shigeo's psychic powers varies over time and can be modeled by the equation $S(t) = 25\sin(t + \arcsin(7/25))$. Similarly, the strength of Teruki's psychic powers can be modeled by the equation $T(t) = 15\cos(t + \arccos(3/5))$. Assuming psychic power is additive, what is the maximum possible combined strength of Shigeo's and Teruki's psychic powers at a single moment?

A. $4\sqrt{97}$ B. $\sqrt{1114}$ C. 20 D. 40 E. NOTA

22. To test ENI-O's matrix processing capabilities, Tadano has devised a simple test. He states a positive integer n, and ENI-O has to come up with a 2×2 matrix with distinct positive integer entries and determinant n. What is the smallest n such that no such matrix exists?

A. 32 B. 59 C. 85 D. 107 E. NOTA

23. Being the only member of the gardening club, Haru loves rose curves (although she doesn't understand the choice of name—they look more like daisies!). She has a polar equation which she *thinks* is a rose curve, but it isn't simplified, so she isn't positive. Help her out: how many petals does the following graph have?

$$r = \sin(2\theta)\cos^2(2\theta) - \sin^3(2\theta) + \cos^2(\theta)\sin(4\theta) - \sin^2(\theta)\sin(4\theta)$$

A. 8 B. 6 C. 12 D. 16 E. NOTA

24. Thanks to her power of clairvoyance, Rei can predict face-down cards with 62% accuracy. Kenji has three cards numbered 1-3. He picks a card at random, where cards 1, 2, and 3 have probabilities $1/6$, $1/3$, and $1/2$ of being chosen, respectively, and places it face-down in front of Rei, who then declares the card has the number 1 on its face. Given this information, what is the probability that the card has the number 1 on its face? Assume that when Rei is wrong, each wrong option is equally likely.

A. $62/157$ B. $31/126$ C. $157/600$ D. $31/50$ E. NOTA

25. Collot has a fair coin which he flips four times, getting n heads. Jack does not see the outcome of the flips but is told to choose an angle θ such that $0 < \theta < \pi/2$. If $\tan(2^n\theta)$ is negative, then Jack wins. Assuming Jack chooses optimally to maximize his odds of winning, what is the probability he wins?

A. $1/2$ B. $5/8$ C. $15/16$ D. $7/8$ E. NOTA

26. One of Legom's eggs can be modeled by the equation $x^2 + y^2 = z(1-z)^{3/2}$ in xyz-space. The maximum possible area of a cross section of the egg taken parallel to the xy-plane can be expressed as $m\pi\sqrt{n}/p$, where m and p are relatively prime positive integers and n is a square-free positive integer. What is $m + n + p$?

A. 141 B. 146 C. 132 D. 11 E. NOTA

27. Emma has to get from her current location to a portal at (1,1,1). However, before she even takes a step, [illegible] distorts space such that any vector $\vec{v}$ becomes $T\vec{v}$ for some 3×3 matrix T. The only information Emma has is that

$$T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\4\\7\end{pmatrix}, \quad T\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix}, \quad T\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}6\\-3\\-1\end{pmatrix}$$

Find the sum of the coordinates of the portal's new location after the distortion.

A. 32 B. 28 C. 54 D. 38 E. NOTA

28. It's Juno's birthday, and everyone is celebrating by wearing birthday hats, which can be modeled by the graph of $2(x^2 + y^2) = z^2$ where $z \leq 0$. However, when Louis was putting on his hat, he accidentally sliced it with his antlers! If the slice can be modeled by the graph of $x + y + z = -1$, which conic section is formed by the intersection of the birthday hat and Louis's slice?

A. Circle B. Non-Circular Ellipse C. Half of a Hyperbola D. Parabola E. NOTA

29. Violet is delivering letters to a street with 30 houses numbered 1-30. She has been told that each house can receive at most one letter, and if house i receives a letter, house $i + 1$ cannot receive a letter (this is not cyclic; houses 1 and 30 can both receive letters). She has also been told that the largest number among any of the houses that receive letters is divisible by 3. Given she delivers at least one letter, how many possible combinations of houses can receive letters? (See problem 17 for information on the F_n notation)

A. $(F_{32} - 1)/2$ B. $(F_{31} - 1)/2$ C. $F_{32} - 1$ D. $F_{31} - 1$ E. NOTA

30. Norman has a magical $\Lambda 7214$ coin that can change its probability of landing heads based on the *value* assigned to a given flip. If the value of a given flip is V, the coin sets its probability of landing heads to $1/(V^2 + 3V + 2)$. Norman flips the coin n times and assigns the value 2^{k-1} to the kth flip for $1 \leq k \leq n$. Let E_n denote the expected sum of the values of the flips that land heads. As n grows large, what value does E_n approach?

A. $7/10$ B. $2/3$ C. $3/4$ D. $3/5$ E. NOTA