Mu Alpha Theta National Convention 2007
Sequences and Series Alpha - SOLUTIONS

1. Common difference = 36 – 10 = 26 **B**

2. Common ratio = $\frac{\pi}{2} \div \frac{1}{2} = \pi$ **A**

3. $a_4 = \frac{2(4)}{4+2} = \frac{4}{3}$ **B**

4. $b_3 = \sin\left(\frac{3 \cdot \pi}{2}\right)$ = -1 **A**

5. We can see that this sequence is increasing exponentially, and the ratio of sequential terms is getting closer to 3. $c_n = 3^n - 2$ fits this model **C**

6. Let $x = \sqrt{30 + \sqrt{30 + \sqrt{30 + ...}}}$, it follows that $x = \sqrt{30 + x}$ → $(x-6)(x+5) = 0$ → x = 6 since it must be a positive number **C**

7.
$$\sum_{k=4}^{9}\left(5k^2 - 2k + 1\right) = \sum_{k=1}^{9}\left(5k^2 - 2k + 1\right) - \sum_{k=1}^{3}\left(5k^2 - 2k + 1\right)$$
$$= \frac{5(9)(9+1)(2(9)+1)}{6} - \frac{2(9)(9+1)}{2} + 9 - \left(\frac{5(3)(3+1)(2(3)+1)}{6} - \frac{2(3)(3+1)}{2} + 3\right) = 1283$$
C

8. Order of differences
$$\begin{array}{ccccc} 1 & & & & \\ & > 2 & & & \\ 3 & & > 1 & & \\ & > 3 & & & \\ 6 & & > 1 & & \\ & > 4 & & & \\ 10 & & > 1 & & \\ & > 5 & & & \\ 15 & & & & \end{array}$$
shows that T(n) is a 2nd degree polynomial

$T(n) = an^2 + bn + c$, solving simultaneously for a, b, c:

$$\begin{array}{l} 1 = a + b + c \\ 3 = 4a + 2b + c \\ 6 = 9n + 3b + c \end{array} \rightarrow a = \frac{1}{2}, b = \frac{1}{2}, c = 0 \rightarrow T(n) = \frac{1}{2}n^2 + \frac{1}{2}n \rightarrow \sum_{n=1}^{7} T(n) = 84$$ **D**

9. 34 – 6 = 28 **E**

n	a_n
1	3
2	4
3	6
4	10
5	18
6	34

10. Rewrite the equation as $a_{k+1} - 3a_k = -2$, giving the characteristic equation $r - 3 = 0$ → r = 3 → $a_k = p \cdot 3^k + q$

$$\begin{aligned}3 &= 3p+q\\ 7 &= 9p+q\end{aligned} \rightarrow \text{p} = \frac{2}{3}\text{, q} = 1 \rightarrow a_k = \frac{2}{3}(3^k)+1 = 2(3^{k-1})+1 \text{. } 2 + 3 - 1 = 4 \ \mathbf{B}$$

11. Total distance = 15 + $10 + \frac{10}{3} + \frac{10}{9} ... = 15 + \frac{10}{1-\frac{1}{3}} = 30$ **B**

12. $2 \cdot \sum_{k=1}^{15} \sin\left(\frac{\pi \cdot k}{4}\right)\cos\left(\frac{\pi \cdot k}{4}\right) = \sum_{k=1}^{15} \sin\left(\frac{\pi \cdot k}{2}\right) = 0$ **B**

13. Let $x = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \ldots}}}}$, then $x = 1 + \frac{1}{3 + \frac{1}{x}} = 1 + \frac{x}{3x+1} \rightarrow$

$3x^2 + x = 4x + 1 \rightarrow 3x^2 - 3x - 1 = 0 \rightarrow x = \frac{3 \pm \sqrt{21}}{6}$.

x must be positive so $x = \frac{3 + \sqrt{21}}{6}$ **C**

14. Solving the equation for K: $50 = K\left(\frac{1}{2}\right)^{t/8}$ when t = 12 gives $K = \frac{50}{\left(\frac{1}{2}\right)^{3/2}} = 100\sqrt{2}$ **E**

15. The last two digits follow a pattern:

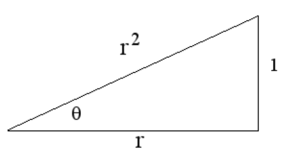
<table>
<tr><td>Number</td><td>Last two digits</td><td rowspan="8">The last digit is always 1, and the second to last goes through the sequence 2, 4, 6, 8, 0, 2, 4, 6, 8, 0, …

So every 5th power gives the same final two digits. Powers of 5, 10, 15, ... 90, 95, 100 will all have 01 as the final two digits.

The last two digits of 21^{100} are 01 C</td></tr>
<tr><td>21</td><td>21</td></tr>
<tr><td>21^2</td><td>41</td></tr>
<tr><td>21^3</td><td>61</td></tr>
<tr><td>21^4</td><td>81</td></tr>
<tr><td>21^5</td><td>01</td></tr>
<tr><td>21^6</td><td>21</td></tr>
<tr><td>21^7</td><td>41</td></tr>
</table>

16. This is the Fibonacci sequence. The next term is the sum of the previous two.
21 + 34 = 55 **C**

17. $499{,}000 = 49{,}900(1-0.55)^{t/30} \rightarrow t = 30\frac{\log(0.1)}{\log(0.45)}$ **B**

18. 

Using the Pythagorean theorem, $1 + r^2 = r^4 \rightarrow$

$r^4 - r^2 - 1 = 0 \to (r^2)^2 - r^2 - 1 = 0 \to r^2 = \frac{1+\sqrt{5}}{2}$

$\sin(\theta) = \frac{1}{r^2} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$ **B**

19. We can see that for $n > 3$, $\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So $|X_3| = \left|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| = 0$ **B**

20. The coefficient of $x^a y^b$ is $\frac{(a+b)!}{a!b!}$, so for $x^{12}y^{13}$ the answer is $\frac{25!}{13!12!}$ **D**

21. $\sum_{n=0}^{k}(n \cdot n!) = (k+1)! \to \sum_{n=1}^{k}(n \cdot n!) = (k+1)! - 1 \to$ **A**
$\sum_{n=2}^{500}(n \cdot n!) = (500+1)! - 1 - 1 = 501! - 2$

22. Rationalizing the denominators, we have
$\left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \left(\sqrt{4}-\sqrt{3}\right) + \ldots + \left(\sqrt{25}-\sqrt{24}\right) = 4$ **D**

23. Only II and III are correct. **E**

24. Observe the sum for small values of n, you will see that $\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{n}{n+1}$

$\sum_{k=1}^{16}\frac{1}{k(k+1)} = \frac{16}{17}$ **B**

25. $\frac{10^9}{9!} = \frac{10^{10}}{10!} \to a_9 = a_{10}$. Only II and III are true. **B**

26. Using partial fraction decomposition, $\frac{2}{k^2+2k} = \frac{1}{k} - \frac{1}{k+2}$. For the series, all terms except for $1 + \frac{1}{2}$ are subtracted out. $\sum_{i=1}^{\infty}\frac{2}{k^2+2k} = \frac{3}{2}$ **C**

27. Order of differences shows that S(n) is a 3rd degree polynomial.

$$\begin{array}{ccccccc} 1 \\ & >4 \\ 5 & & >6 \\ & >10 & & >3 \\ 15 & & >9 \\ & >19 & & >3 \\ 34 & & >12 \\ & >31 & & >3 \\ 65 & & >15 \\ & >46 \\ 111 \end{array} \rightarrow \begin{aligned} 1 &= a+b+c+d \\ 5 &= 8a+4b+2c+d \\ 15 &= 27a+9b+3c+d \\ 34 &= 64a+16b+4c+d \end{aligned} \rightarrow a = \frac{1}{2}, b = 0, c = \frac{1}{2}, d = 0$$

$S(n) = \frac{n^3}{2} + \frac{n}{2}$. $S(9) - S(8) = 109$ **C**

28. Let d_A, d_B be the common differences of the two sequences. Then $a_{11} = a_1 + 10d_A$ and $b_{11} = b_1 + 10d_B \rightarrow \frac{a_{11}}{b_{11}} = \frac{a_1 + 10d_A}{b_1 + 10d_B}$

Using the formula for the sum of an arithmetic series:

$(4n+27)\left(\frac{n}{2}\left(2a_1+(n-1)d_A\right)\right) = (7n+1)\left(\frac{n}{2}\left(2b_1+(n-1)d_B\right)\right) \rightarrow$

$\frac{\left(\left(2a_1+(n-1)d_A\right)\right)}{\left(\left(2b_1+(n-1)d_B\right)\right)} = \frac{(7n+1)}{(4n+27)} \rightarrow$ Using n = 21 gives the ratio we are looking for:

$\frac{a_1+10d_A}{b_1+10d_B} = \frac{(7(21)+1)}{(4(21)+27)} = \frac{4}{3}$ **B**

29. Let a be the first number and b be the last number. a is 4005. We can see that b + 9 = 5004, so it follows that n = $\frac{5004-4005}{9} = 111$ **B**

30. $2 = x^{x^{x^{x^{\cdots}}}} \rightarrow 2 = x^2 \rightarrow x = \sqrt{2}$ **A**