Interschool Test Solutions
FAMAT State Convention 2002

1. The first number is the sum of the first three prime numbers (2+3+5=10). The second number is the sum of the second, third, and fourth prime numbers (3+5+7=15). The pattern continues for more prime numbers. The next two terms in the sequence would be 23+29+31=83 and 29+31+37=97.

2. (4/5)(3/5)(4/5)(1)=48/125

3. 1.9:1

4. 6 PM April 19, 2002

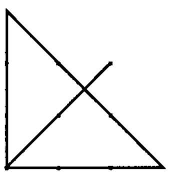
5. Sides of the triangle: 3, (2)(3)(b)/(3+b), and b The perimeter is 10, so 3+b+(6b)/(3+b)=10. Solving for b gets two solutions, with only one being positive... $b=-1+\sqrt{22}$. Thus the middle length is $8-\sqrt{22}$. Thus the area of the triangle is $\sqrt{5(5-3)(5-8+\sqrt{22})(5+1-\sqrt{22})}=\sqrt{90\sqrt{22}-400}$

6. 2008

7. The roots of this equation are $\pm\sqrt{3}$ and $\pm 7/3$. The closest root is thus $\sqrt{3}$. Newton's method has:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{9x_n^4-76x_n^2+147}{36x_n^3-152x_n}$$

The 18th iteration gives a solution that is .000128 away from $\sqrt{3}$, wheras the 19th iteration gives a solution that is .000080 away from $\sqrt{3}$.

8. 4, 

9. The perimeter of an n-sided regular polygon circumscribing a circle of radius r is 2nrtan(180°/n). The perimeter of an n-sided regular polygon inscribed in a circle of radius r is 2nrsin(180°/n). Thus, the ratio of the two would be sec(180°/n). In this case n equals 20 and the answer is 9.00 (to the nearest hundredth).

10. This is an arithmetic-geometric series with a=9, d=9, and r=1/3. By the formula: $a+(a+d)r+(a+2d)r^2+\ldots=a/(1-r)+(rd)/(1-r)^2$, we get a solution of 81/4 or 20.25.

11. There are 13 tiles that have a blank on them (blank to 12). There are also 13 tiles that have a one on them, but you have already accounted for the one with the blank on it, so 12. Similarly you can go through the rest of the tiles to come up with 13+12+11+...+2+1=91.

12. $$\int_2^4\frac{7x^2-x-66}{2x^3-9x^2-8x+15}dx=\int_2^4\frac{-3dx}{2x+3}+\int_2^4\frac{2dx}{x-5}+\int_2^4\frac{3dx}{x-1}=$$
$$-\frac{3}{2}\ln|2x+3|\Big|_2^4+2\ln|x-5|\Big|_2^4+3\ln|x-1|\Big|_2^4=(3/2)\ln(7/11)+\ln 3$$

13. An algebraic number is a number that is a root of a polynomial equation $a_0x^n+a_1x^{n-1}+a_n=0$, where the coefficients are integers. In this case A, B, D, and F are each potential algebraic numbers. C and F are transcendental.

14. 2:1 odds means a horse has a 1/3 chance of winning; 3:1 odds, a 1/4 chance; and 5:1 odds, a 1/6 chance. Adding these probabilities up and subtracting from 1 gives the chance for the last horse to win, 1/12. So in the same fashion as the odds were given, 11:1.

15. If you get 5 one's on your first roll, then that counts as a Yahtzhee, 50 points. If you roll another 5 one's, then you get 5 points, plus the 100 point bonus for another yahtzhee. Similarly, going through all the digits on the die and adding the 35 point bonus gets you a total of 790. You also need a three and a four of a kind. If you roll 5 six's on each roll, then you get a score of 30+100+30+100. Next is the full house. Rolling a Yahtzee on this will get you 25+100 points. Next are the straights. Rolling a Yahtzhee on these allows you to put your score down with full value as a joker. The 4-straight is worth 30 points and the 5-straight is worth 40 points. Finally, if you roll 5 6's on the chance roll, you would have a score of 30+100, total 1575.

16. 16 hours. 6:30 AM to 10:30 PM on day 1, then 2:30 PM on day 2, then repeat the cycle.

17. Ovals of Cassini

18. ♩ = 90 means that there would be 90 quarter notes every minute. If there are 4 quarter notes in each measure, and 96 measures in the piece, then there are a total of 384 beats in the music. Dividing this value by 1.5 quarter notes per second gives that the length of the song is 256 seconds long.

19. Use dimensional analysis:

$$\frac{1000\ \text{lb—ft}^2}{1\ \text{s}^2}\ \Big|\ \frac{(.3048\ \text{m})^2}{(1\ \text{ft})^2}\ \Big|\ \frac{0.45359\ \text{kg}}{1\ \text{lb}} = 42.14\ \text{kg m}^2\ \text{s}^{-2} = 42.14\ \text{Joules}$$

20. 6, 12, 124

21. The law of tangents for a plane triangle *MAT* has that tan[0.5(*M*+*A*)]/tan[0.5(*M*-*A*)]=(*m*+*a*)/(*m*-*a*), in this case, 39/17.

22. 15 points (love 40 to win 5 points+7 points in tiebreaker + three in the next game)

23. π/2 , this is called Wallis' product.

24. This is just a Pascal's triangle problem. The first Hanning filter gets you to the second row of Pascal's triangle, 1 2 1. The second filter gets you to the fourth row, 1 4 6 4 1. The sixth filter gets you to the twelth row, 1 12 66 220 495 792 924 792 495 220 66 12 1. So the weight of the center point is 924/4096, which when simplified is 231/1024.

25. Parking Lot

26. 0.25

27. 44 (all numbers have 6 factors)

28. This is essentially a large cosine law problem. You set up a triangle with one side equal to a radius of the earth plus 50 meters. Another side has length equal to 150 km. The angle between those two sides is 90.5 degrees. The third side has the radius of the earth plus the height above the earth you are looking for. So the problem sets itself up as:

$$(r_e+x)^2=(r_e+50)^2+(150000)^2-2(r_e+50)(150000)\cos(90.5°)$$

Substituting for r_e equal to 6370000 meters and solving for x gives 3124 meters

29. The spectral radius of a matrix is equal magnitude of the largest eigenvalue. The eigenvalues are the λ's that solve the equation:

$$\begin{vmatrix} 4-\lambda & 21 & -5 \\ 0 & 0-\lambda & 1 \\ 2 & 3 & 6-\lambda \end{vmatrix} = 0$$

, which reduces to $\lambda^3-10\lambda^2+31\lambda-30=0$, which has roots of 2, 3, and 5. Thus the spectral radius is 5.

30. All the zeroes at the end of 2002! will come from all products of 2 and 5. Because there are plenty of factors of 2 in 2002!, we need only to count the number of factors of 5. Every multiple of 5 has 1 five in it, every multiple of 25 has 2 fives in it, every multiple of 125 has 3 fives in it, and every multiple of 625 has 4 fives in it. There are 400 multiples of five in the factors of 2002!. There are 80 multiples of 25 in the factors of 2002!, but we have already accounted of the number of 5's in these multiples, so we need only add 80 to our list (400+80). There are 16 multiples of 125 in the factors of 2002!. Again, we only need to account for one set of 5's in this list (400+80+16). There are 3 multiples of 625 in the factors of 2002. Total: 499

31. The time from when the ball is thrown to when it is caught is approximately for 1.633 seconds (using gravity). After this point, it could be anywhere at a radius of (1.633 s)(30 m s^{-1})=48.99 m from the quarterback. If we set up a coordinate system where the quarterback is at the center, then the circle is described by $x^2+y^2=(48.99\text{ m})^2$. The wide receiver's position is a function of time. Let's work up from his acceleration:

$$a=\begin{cases}8-3t & t\le 8/3\\ 0 & t>8/3\end{cases}\ \Rightarrow\ v=\begin{cases}8t-(3/2)t^2 & t\le 8/3\\ 32/3 & t>8/3\end{cases}\ \Rightarrow$$

$$x=\begin{cases}20 & t\le 2\\ 20-\left(4t^2-0.5t^3-12\right)\sin 30° & 2<t\le 8/3\\ 446/27-(32/3)\sin 30°(t-8/3) & t>8/3\end{cases}$$

$$y=\begin{cases}4t^2-0.5t^3 & t\le 2\\ 12+\left(4t^2-0.5t^3-12\right)\cos 30° & 2<t\le 8/3\\ 12+94\sqrt{3}/27+(32/3)\cos 30°(t-8/3) & t>8/3\end{cases}$$

The time frame we are looking for is after 8/3 seconds. So if we take that part of the above position and plug it into our caught football's position, we can solve for t in the quadratic equation that results. In this case, t is 6.015 seconds. However, the question asks how long after the snap before the quarterback should throw the ball, or 6.015 -1.633 seconds = 4.4 seconds.

32. $m/(2^{m-1})$

33. Simplify to: $\pi^{\pi^{x^3}} = 3$. Take two logs to get: $x^3 \log\pi = \log(\log 3) - \log(\log\pi)$. Thus, x=-.33.

34. What is the result of 7 multiplied by 8 multiplied by 9 divided by 4? 126

35. What is the product of 20 times 17 divided by 4? 85

36. What is the sum of 70 and 10 times 6? 130

37. What is the result of 2002 minus 2001? 1

38. What is the product of 2002 and 0? 0

39 & 40. The best solution I found for solving this problem is to put the times and constraints in a computer to solve for the length of time required of each segment, and how fast the car would be going at the turning points. Three paths came out being close to each other in time. They were A to D to F to K (70.19 minutes), A to B to F to K (70.22 minutes), and A to E to F to K (70.71 minutes). So the answer to question 15 is 70 minutes. By the way, the length of road AE was 64.27 km ($2\pi/4*25$ km+25 km), and the length of road FK was 46.00 km (solved by using calculus to find the length of a given curve).

41. The smallest possible value for the expression is when P is the centroid of an equilateral triangle, a value of 8.

42. 2 lollipops, 3 balloons, and 2 packs of gum

43. $3\pm2i$, 19/13, and -17/19

44. The roots are 2002 cis[n(360/2002)], where n goes from 1 to 2002. If you take the product of all these roots you get

$$2002^{2002}\text{cis}\left[(360/2002)+2(360/2002)+\cdots 2002(360/2002)\right], \text{ or}$$

$$2002^{2002}\text{cis}\left[(360/2002)(2002\times2003/2)\right] = 2002^{2002}\text{cis}(180) = -2002^{2002}.$$

45. The sum of digits in the numbers 1 to 9 is 45; for 10 to 19 the sum is 10+45; for 20 to 29, 20+45…
for 1 to 99, 450+10(45)=900. For the numbers 100 to 199, just add 100 1's to the 900=1000. So to get from pages 1 to 999, you have 4500+10(900) =13500. Adding in another 100 gets you, 1 to 1099=100+900+13500=14500. Solving for the remainder of the sum gets you to 1182 pages.

46. Niccolo Tartaglia of Brescia, he was hit in the head with an axe by the French army during an attack on his home town when he was young. His mother nursed him back to health.

47. There are six 'shortest' directions the ant can go. It can either start on the 2 by 5 side, the 3 by 5 side, or the 2 by 3 side. The 'shortest' distance it can travel is by taking the side it starts on and adjoining it to the adjacent side that contains the point B (two choices for each starting side). If you combine the two sides and flatten them out, you will have a larger rectangle. The shortest distance to travel is a diagonal of this longer rectangle. Choosing the shortest path of the six distances gives a distance of $5\sqrt{2}$

48. 2560

49.

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   1  6  3  5  = 15
      8  2  9  = 19
+        4  7  = 11
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   2  5  1  1
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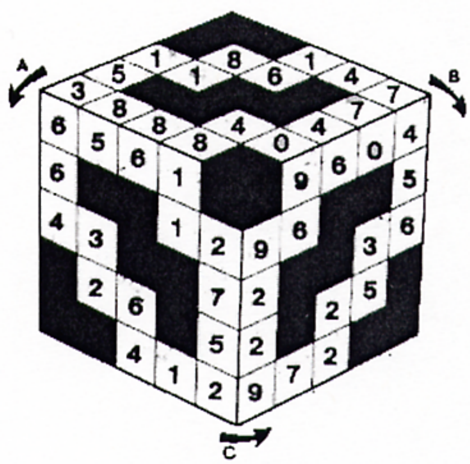

50.