Mu Alpha Theta National Convention: Denver 2001
Limits & Derivatives Topic Test – Solutions

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1. **(A)**. The given limit is the definition of the derivative applied to $f(x) = x^2$.

2. **(B)**. Differentiating, we get $y' = -18x^2 + 5x^4 + 12x^3$ so the slope of the tangent line at $x = 2$ is $-18(2)^2 + 5(2)^4 + 12(2)^3 = 104$. Thus, the equation is $y - 33 = 104(x - 2)$ or $(y + 175)/104 = x$.

3. **(C)**. The function will be strictly increasing when the derivative is positive. Solving the inequality $3z^2 - 30z + 48 > 0$ yields $z < 2$ or $z > 8$.

4. **(A)**. By direct substitution, the answer is $(3)^5 - 2(3)^2 + 6 = 231$.

5. **(A)**. The denominator of a_n grows much faster than the numerator so $\lim_{n\to\infty} a_n = 0$.

6. **(C)**. Recalling the difference of perfect cubes formula $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, the expression simplifies to $2r^2 + 1 = t^2 - tr + r^2$. Differentiating implicitly yields

$$4r\frac{dr}{dt} = 2t - r - t\frac{dr}{dt} + 2r\frac{dr}{dt}$$

which makes $\dfrac{dr}{dt} = \dfrac{2t - r}{t + 2r}$.

7. **(B)**. By the Product and Chain rules, $H'(x) = 2f'(2x)g(x) + g'(x)f(2x)$, meaning $H'(2) = 2f'(4)g(2) + g'(2)f(4)$. Substituting the necessary values from the table, we get $H'(2) = 2(7)(3) + (-6)(8) = 42 - 48 = -6$.

8. **(A)**. Again, by the Product and Chain rules, $(f(h(x))g(x))' = h'(x)f'(h(x))g(x) + g'(x)f(h(x))$. Letting $x = 1$ and using the fact that $h(1) = 4$, we get $h'(1)f'(4)g(1) + g'(1)f(4) = (2)(7)(5) + (8)(8) = 70 + 64 = 134$.

9. **(E)**. Multiple applications of the Chain Rule produces

$$\frac{d}{dx}f(g(h(x))) = h'(x)g'(h(x))f'(g(h(x)))$$

Thus, the answer is $h'(1)g'(h(1))f'(g(h(1))) = h'(1)g'(4)f'(1) = (2)(-1)(-1) = 2$.

10. **(B)**. $(\sec p)' = \sec p \tan p$. Because c is in the second quadrant

$$\sec c = \frac{-1}{\sqrt{1 - \sin^2 p}} = -\frac{5}{3} \qquad \text{and} \qquad \tan p = \frac{-\sin p}{\sqrt{1 - \sin^2 p}} = -\frac{4}{3}$$

Letting $p = c$, we get $\sec c \tan c = (-5/3)(-4/3) = 20/9$.

11. **(C)**. If we denote the leg of the triangle by s, then the area of the triangle is $A = s^2/2$. Taking the derivative with respect to time t yields

$$\frac{dA}{dt} = s\frac{ds}{dt} = \left(\frac{2\sqrt{6}}{\sqrt{2}}\right)(-3) = -6\sqrt{3}$$

Thus, the area of the triangle is shrinking at a rate of $6\sqrt{3}$ square inches per minute or $\sqrt{3}/10$ square inches per second.

12. **(A)**. As $x \to 0$, $u \to -\infty$. Thus, $\lim_{x\to 0}\left(x^2 + \frac{1}{x}\right)^x = \lim_{u\to\ \infty}\left(u + \frac{1}{u^2}\right)^{\frac{1}{u}}$.

13. **(B)**. The cosine and secant functions are reciprocals of each other so $y = \sec^3 x \cos^3 x = 1$, making the 500th derivative of y equal to zero.

14. **(D)**. The acceleration is given by the second derivative of the position function. Thus

$$y'(t) = \sin 2t + 2t\cos 2t$$
$$y''(t) = 2\cos 2t + 2\cos 2t - 4t\sin 2t = 4\cos 2t - 4t\sin 2t$$

Therefore, $y''(\pi) = 4\cos 2\pi - 4\pi\sin 2\pi = 4(1) - 4\pi(0) = 4$.

15. **(A)**. By the Chain Rule, $2P'(2n+6) = 4n + 18$ or $P'(2n+6) = 2n + 9$. Letting $n = (n-6)/2$, we get $P'(2(n-6)/2 + 6) = P'(n) = 2(n-6)/2 + 9 = n + 3$.

16. **(D)**. Differentiating twice, we get

$$y' = 2v\cos v^2$$
$$y'' = 2\cos v^2 - 2v(-2v\sin v^2) = 2\cos v^2 - 4v^2\sin v^2$$

17. **(C)**. Denote the radius of the sector by r and the central angle made by the radii to be θ. The perimeter of the sector is then $r\theta + 2r = 4$ and the area $A = \theta r^2/2$. Solving for θ in the first equation and substituting this into A produces

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{4-2r}{r}\right) = 2r - r^2$$

so $A' = 2 - 2r$, which has a critical value of $r = 1$. By the First Derivative Test, this value yields a global maximum so the desired radius is 1 inch.

18. **(B)**. $\frac{d}{d\theta}\sin^2\theta = 2\sin\theta\cos\theta = \sin 2\theta$ so $a = 2$.

19. **(D)**. Setting the first and second derivatives of y equal to zero yields

$$4x^3 - 36x^2 + 96x - 64 = 0 \to x \in \{1, 4\}$$
$$12x^2 - 72x + 96 = 0 \to x \in \{2, 4\}$$

The only common solution is $x = 4$. It's easy to check that the sign of the second derivative switches sign around this value. Thus, $(a, b) = (4, 1)$. By the First Derivative Test, $x = 1$ is a global minimum so $(c, d) = (1, -26)$, making $ac - bd = 4 - (-26) = 30$.

20. **(A)**. Since $\dfrac{\cos A \cos C - \sin A \sin C}{\cos C \sin A + \sin C \cos A} = \dfrac{\cos(A+C)}{\sin(A+C)} = \cot(A+C)$, the limit equals $\cot 2A$.

21. **(B)**. Differentiating implicity, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

Thus, the slope of the tangent line is $-\sqrt{9/1} = -3$ and the equation is $y-9 = -3(x-1) \rightarrow y = -3x+12$. This line has an x-intercept of $(4, 0)$ and y-intercept of $(0, 12)$ so $a + b = 4 + 12 = 16$.

22. **(D)**. Using the change-of-base formula, $\log_2 x = (\ln x)/(\ln 2)$; thus, $y(x) = (\ln x)/(x \ln 2)$. By the Quotient Rule, $y'(x)$ is

$$\frac{1}{\ln 2}\left(\frac{(1/x)(x) - (1)(\ln x)}{x^2}\right) = \frac{1 - \ln x}{x^2 \ln 2}$$

Thus, $y'(4) = (1 - \ln 4)/(16 \ln 2) = (1 - 2\ln 2)/(16 \ln 2) = (1 - 2a)/(16a)$.

23. **(A)**. Factoring gives us $\displaystyle\lim_{x\to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x - 1) = 2$.

24. **(B)**. Implicit differentiation produces

$$2xy + x^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy = 0 \rightarrow \frac{dy}{dx} = -\frac{2y}{x}$$

The answer is $-2(3)/(1) = -6$.

25. **(D)**. For B to be continuous, $2(1) - (1)^2 = (1)^2 + k(1) + p$, or $k + p = 0$. If B is to be differentiable, $2 - 2(1) = 2(1) + k$ so $k = -2$. Thus, $p = 2$ and $(k, p) = (-2, 2)$.

26. **(C)**. By the Quotient Rule, $\dfrac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1 + \cos x}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x}$.

27. **(A)**. The equation of the tangent lines are $y - 11 = (2(1) + 6)(x - 1) \rightarrow y = 8x + 3$ and $y - 5 = (e^0)(x - 0) \rightarrow y = x + 5$. Setting y values equal to each other, we get $x = 2/7$; thus, $y = 2/7 + 5 = 37/7$.

28. **(E)**. As $\alpha \rightarrow \pi/2^+$, $\arctan(\tan\alpha) \rightarrow -\infty$. However, as $\alpha \rightarrow \pi/2$, $\arctan(\tan\alpha) \rightarrow \infty$. These limits aren't equal so the limit does not exist.

29. **(C)**. $y' = 6x + \dfrac{4}{x^2} \rightarrow y'(2) = 12 + 1 = 13$.

30. **(C)**. Applying L'Hôpital's Rule several times, we get

$$\lim_{t\to 0}\frac{7t^2 + 14\cos t - 14}{t^4} = \lim_{t\to 0}\frac{14t - 14\sin t}{4t^3} = \lim_{t\to 0}\frac{7 - 7\cos t}{6t^2} = \lim_{t\to 0}\frac{7\sin t}{12t} = \frac{7}{12}$$

Thus, $m^2 + n^3 = 49 + 1728 = 1777$.

31. **(B)**. The Maclaurin series for $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ so the series for $z(r) = \sin r^2$ is $\sum_{n=0}^{\infty} \frac{(-1)^n (r^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (r)^{4n+2}}{(2n+1)!}$. Notice that the coefficient of the r^3 term of this series is 0. Consequently, $z^{(3)}(0)/3! = 0$ or $z^{(3)}(0) = 0$.

32. **(A)**. Recall that $f(x_0 + c) \approx f(x_0) + cf'(x_0)$. Letting $f(x) = x^{1/5}$, $x_0 = 32$, and $c = 1$, we get $f(33) \approx 32^{1/5} + (1)(1/5)(32)^{-4/5} = 2 + 1/80$.

33. **(C)**. Those familiar with the properties of the normal curve knows that it has inflection points at one standard deviation (σ) away from the mean (μ). Otherwise, we shall derive this property from scratch by setting $y'' = 0$:

$$y' = \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \rightarrow y'' = \frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right) = 0$$

The second derivative will equal zero if $(x - \mu)^2 = \sigma^2$ or $x = \mu \pm \sigma$.

34. **(A)**. Let r be a critical value of $f(x)$. Notice that $f(x-2)$ is a function with the property we want because $f'((r+2)-2) = f'(r) = 0$. After some computationally intensive algebra (or the synthetic division trick), we find that $f(x-2) = 3x^5 - 30x^4 + 95x^3 - 90x^2 - 2017$. The constant term is irrelevant as it vanishes upon differentiation so we may replace it with any other value to try to match the answer choices. In particular, a constant term of 1 gives the function in choice A.

35. **(D)**. $\lim_{x\to\infty} \frac{5x^2 - 3x + 1}{2x^2 + 17} = \lim_{x\to\infty} \frac{5x^2}{2x^2} = \frac{5}{2}$.

36. **(C)**. Careful! We want the rate of change of the rate of change of the volume, or d^2V/dt^2! For a sphere, $V = 4\pi r^3/3$. Thus

$$\begin{aligned} \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} = 8\pi r^2 \\ \frac{d^2V}{dt^2} &= 16\pi r \frac{dr}{dt} = 16\pi(6)(2) = 192\pi \end{aligned}$$

37. **(B)**. Let $\lim_{n\to\infty} a_n = L$. Since $\lim_{n\to\infty} a_{n-1}$ also equals L, we have $L = 2/(L+2)$. Solving this equation for the positive root, we get $L = -1 + \sqrt{3}$.

38. **(B)**. Taking the natural log of both sides of the equation produces

$$\begin{aligned} \ln y &= y \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \frac{dy}{dx} \ln x + \frac{y}{x} \end{aligned}$$

Solving for dy/dx gives the answer in choice B.

39. **(D)**. Rewrite the expression as

$$\left(\frac{(3x^2+6x-1)(x^2+4x+3)-(2x+4)(x^3+3x^2-x-3)}{(x^2+4x+3)^2}\right)(x^2+4x+3)$$

Notice the first quantity in parenthesis is the derivative of

$$\frac{x^3+3x^2-x-3}{x^2+4x+3}=\frac{(x+3)(x+1)(x-1)}{(x+3)(x+1)}=x-1$$

which is 1. So the answer is simply x^2+4x+3.

40. **(A)**. Using L'Hôpital's Rule twice with respect to c, we get

$$\lim_{c\to 0}\frac{2h'(x+2c)-2h'(x+c)}{2c}=\lim_{c\to 0}\frac{2h''(x+2c)-h''(x+c)}{1}=h''(x)$$

which is just the second derivative of h. Thus, the limit equals $((1-x^2)\ ^{1/2})'=(-1/2)(1-x^2)\ ^{3/2}(-2x)=x/\sqrt{(1-x^2)^3}$.