ARML Power Contest – November 2001 – Cevians

The Solutions

1. Using Stewart's Formula, $a^2m + b^2n = x^2c + mnc$, let $m = n = \frac{c}{2}$. Then $a^2\frac{c}{2} + b^2\frac{c}{2} = x^2c + \frac{c}{2}\frac{c}{2}c \Rightarrow$

$$\frac{a^2}{2} + \frac{b^2}{2} = x^2 + \frac{c^2}{4} \Rightarrow \frac{2a^2 + 2b^2 - c^2}{4} = x^2 \Rightarrow \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2} = x$$

2. Area of the triangle. $K = \frac{1}{2}cx \Rightarrow x = \frac{2K}{c}$.

Area of the triangle. $K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$

$$K = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)}$$

$$K = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)}$$

$$K = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{4}$$

$$x = \left(\frac{2}{c}\right)\frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{4}$$

$$x = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{2c}.$$

3. By the Angle Bisector Theorem, $\frac{m}{n} = \frac{b}{a}$ and $m + n = c$. Therefore, $m = \frac{bc}{a+b}$ and $n = \frac{ac}{a+b}$. Plugging these into Stewart's Formula yields: $a^2\frac{bc}{a+b} + b^2\frac{ac}{a+b} = x^2c + \frac{bc}{a+b}\frac{ac}{a+b}c \Rightarrow \frac{abc(a+b)}{a+b} = x^2c + \frac{abc^3}{(a+b)^2} \Rightarrow$

$$\Rightarrow ab - \frac{abc^2}{(a+b)^2} = x^2 \Rightarrow \frac{ab(a+b)^2 - abc^2}{(a+b)^2} = x^2 \Rightarrow \frac{ab((a+b)^2 - c^2)}{(a+b)^2} = x^2 \Rightarrow \frac{\sqrt{ab(a+b-c)(a+b+c)}}{a+b} = x.$$

4. $AM = MB$ and so $AH + HM = HB - HM$.

Therefore,

$h \tan x + h \tan 2x = h \tan 3x - h \tan 2x$

$\tan x + 2\tan 2x = \tan(x + 2x)$

$$\tan x + 2\tan 2x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$$

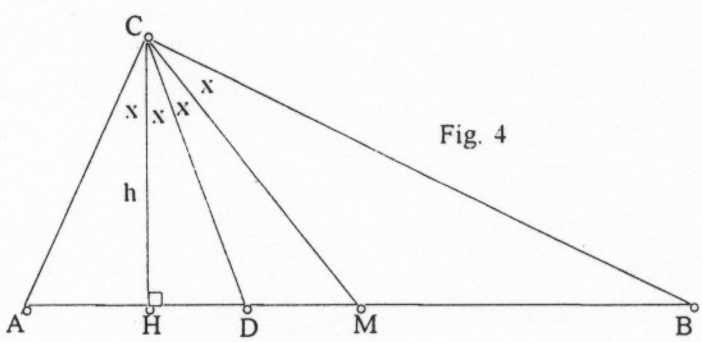


Fig. 4

$\tan x - \tan^2 x \tan 2x + 2\tan 2x - 2\tan x \tan^2 2x = \tan x + \tan 2x \Rightarrow$

$-\tan^2 x \tan 2x + \tan 2x - 2\tan x \tan^2 2x = 0 \Rightarrow \quad -\tan^2 x + 1 - 2\tan x \tan 2x = 0 \Rightarrow$

$-\tan^2 x + 1 - \dfrac{4\tan^2 x}{1-\tan^2 x} = 0 \Rightarrow \quad \tan^4 x - 6\tan^2 x + 1 = 0 \Rightarrow \quad \tan x = \sqrt{3 \pm 2\sqrt{2}} \Rightarrow \quad x = \tan^{-1}\sqrt{3 \pm 2\sqrt{2}} \Rightarrow$

$x = 22.5°$ *or* $67.5° \Rightarrow x = 22.5°$. Therefore, $\angle A = 67.5°$, $\angle B = 22.5°$, and $\angle C = 90°$.

5. By Ceva's Theorem, $\dfrac{AD}{DB} \cdot \dfrac{BM}{MC} \cdot \dfrac{CH}{HA} = 1$. But since M is a midpoint, $\dfrac{BM}{MC} = 1$ and since $\overline{CD}$ is an angle bisector, $\dfrac{AD}{DB} = \dfrac{b}{a}$. Therefore, $\dfrac{CH}{HA} = \dfrac{a}{b}$. Let $CH = at$ and $HA = bt$, then $at + bt = b$, or $t = \dfrac{b}{a+b}$.

By the Pythagorean Theorem, $BH^2 = AB^2 - HA^2$, so $BH^2 = c^2 - b^2t^2$ and $BH^2 = BC^2 - CH^2$, so $BH^2 = a^2 - a^2t^2$.

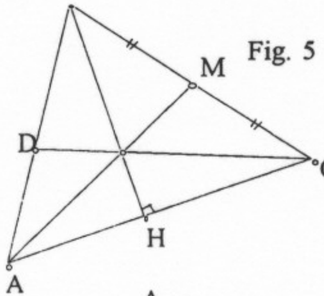


Fig. 5

Therefore, $c^2 - b^2t^2 = a^2 - a^2t^2$ and $t^2 = \dfrac{a^2-c^2}{a^2-b^2}$. Substituting for t, $\left(\dfrac{b}{a+b}\right)^2 = \dfrac{a^2-c^2}{a^2-b^2}$ and so $\dfrac{b^2}{a^2-c^2} = \dfrac{a+b}{a-b}$.

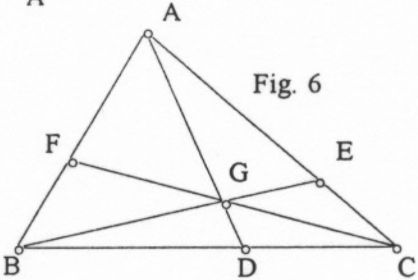


Fig. 6

6. Let s = the semi perimeter of ΔABC, then AE = $s - c$, EC = $s - a$, CD = $s - b$, DB = $s - c$, BF = $s - a$, and FA = $s - b$.

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{s-c}{s-a} \cdot \frac{s-b}{s-c} \cdot \frac{s-a}{s-b} = 1.$$

Therefore, the cevians are concurrent. This is known as **Geronne's Point.**

7. ΔDEF is the orthic triangle of ΔABC, with P the ortho center, the intersection of the altitudes. $\Delta AFP \approx \Delta CDP$ and so $\angle FAP \cong \angle DCP$. Since quadrilateral $AEPF$ has two right angles it must be cyclic. The same is true for quadrilateral $CDPE$.

Because they intercept the same arc, $\angle FAP \cong \angle FEP$ and $\angle DCP \cong \angle PED$. Therefore, $\angle FEP \cong \angle PED$.

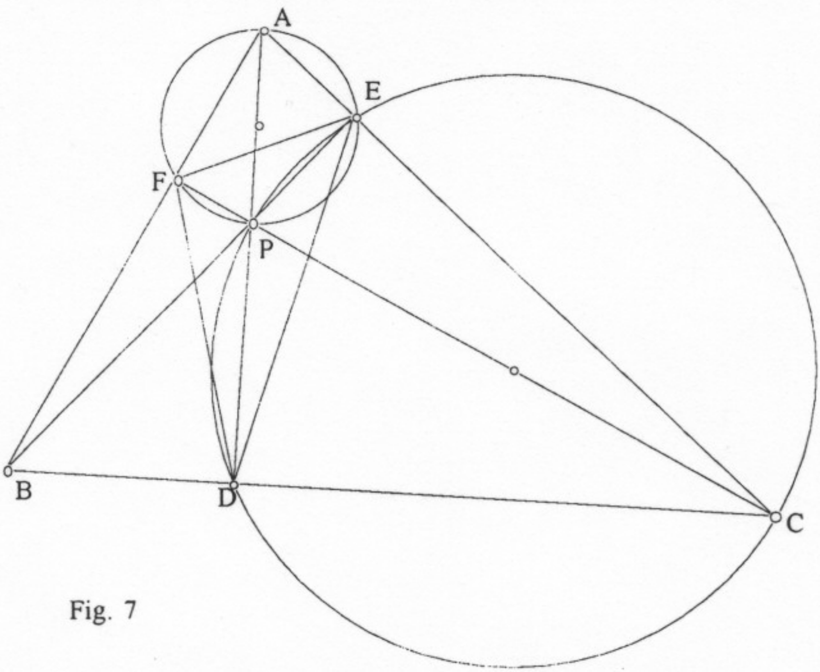


Fig. 7

The same reasoning can be used to show $\angle EDP \cong \angle FDP$ and $\angle DFP \cong \angle EFP$ and so $\overline{DA}$, $\overline{EB}$, and $\overline{FC}$ are bisectors of the angles of ΔDEF and so P is the incenter of ΔDEF.

8. By Hero's Formula the area of ΔABC is $K_{ABC} = \sqrt{s(s-14)(s-25)(s-\sqrt{149})}$ where $s = \dfrac{\sqrt{149}+39}{2}$.

$$= \sqrt{\left(\frac{\sqrt{149}+39}{2}\right)\left(\frac{\sqrt{149}+11}{2}\right)\left(\frac{\sqrt{149}-11}{2}\right)\left(\frac{39-\sqrt{149}}{2}\right)} = \frac{\sqrt{(39^2-149)(149-11^2)}}{4} = \frac{\sqrt{28456}}{4} = 49$$. Using

Routh's Theorem, the area of ΔPQR is $K_{PQR} = \dfrac{\left(\frac{3}{5}-1\right)^2}{\left(\frac{3}{5}\right)^2+\frac{3}{5}+1} \cdot K_{ABC} = \left(\frac{4}{49}\right)(49) = 4$.

9. Using Routh's Theorem for the inscribed cevian triangle,

$$28 = \frac{(1)(4)(7)}{(2)(5)(8)} \cdot \Delta ABC \quad \Rightarrow \quad 28 = \frac{28}{80} \cdot \Delta ABC \quad \Rightarrow \quad 80 = \Delta ABC$$

Let K = the area of the interior cevian triangle. $K = \dfrac{(28-1)^2}{(28+4+1)(7+7+1)(4+1+1)} \cdot 80 = \dfrac{216}{11}$.

10.
$$K_{PQR} = \frac{(n^3-1)^2}{(n^2+n+1)(n^2+n+1)(n^2+n+1)} \cdot K_{ABC}$$
$$\frac{K_{PQR}}{K_{ABC}} = \frac{\left((n-1)(n^2+n+1)\right)^2}{(n^2+n+1)^3}$$
$$\frac{K_{PQR}}{K_{ABC}} = \frac{(n-1)^2}{(n^2+n+1)} = \frac{(n^2-2n+1)}{(n^2+n+1)}$$

11. Let cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CE}$ be concurrent at point P and let Q be the intersection $\overline{AD'}$ *and* $\overline{BE'}$, the isogonal conjugates of $\overline{AD}$ and $\overline{BE}$. Let the distances from P to sides $\overline{BC}$, $\overline{AC}$, and $\overline{BC}$ be a', b', and c', respectively, and the distances from Q to the sides $\overline{BC}$, $\overline{AC}$, and $\overline{BC}$ be a'', b'', and c'', respectively. Using $\overline{AD}$ *and* $\overline{AD'}$ and The Isogonal Theorem, $\frac{b'}{c'} = \frac{c''}{b''}$. Using $\overline{BE}$ and $\overline{BE'}$ and The Isogonal Theorem, $\frac{c'}{a'} = \frac{a''}{c''}$. Combining the two equations using multiplication yields

$$\frac{b'}{c'} \cdot \frac{c'}{a'} = \frac{c''}{b''} \cdot \frac{a''}{c''}$$
$$\frac{b'}{a'} = \frac{a''}{b''}$$

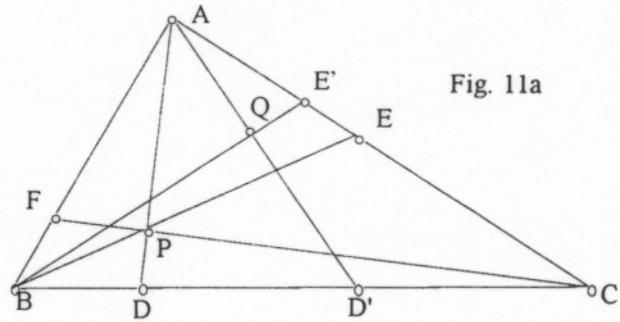


Fig. 11a

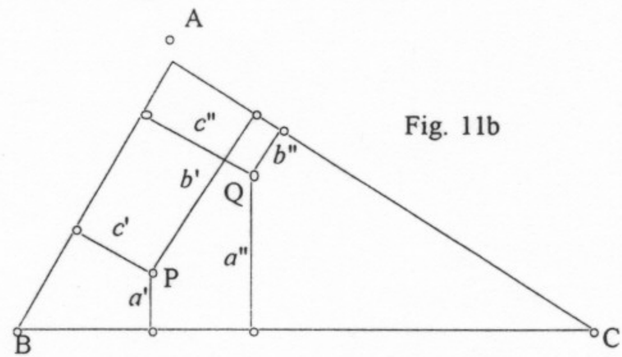


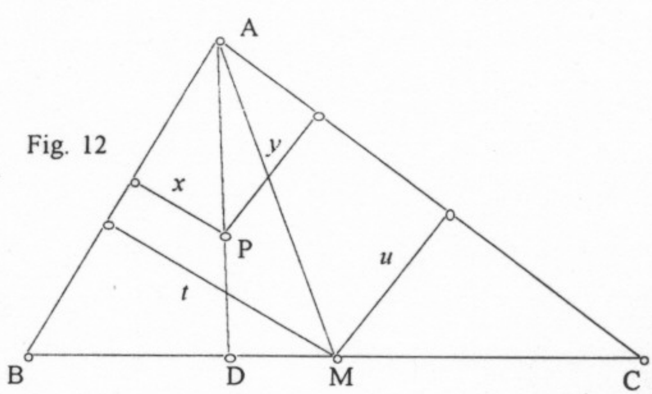
Fig. 11b

Let $\overline{CF'}$ be a cevian from C through point Q. By The Isogonal Theorem, $\overline{CF}$ *and* $\overline{CF'}$ must be isogonal. Therefore, isogonals of concurrent cevians are concurrent.

12. $\Rightarrow$ Assume P is a point on symmedian $\overline{AD}$. By the Isogonal Theorem, $\frac{x}{y} = \frac{u}{t}$. Because M is a midpoint, $K_{ABM} = K_{ACM}$ and $\frac{1}{2}ct = \frac{1}{2}bu$. Therefore, $\frac{c}{b} = \frac{u}{t}$ and so $\frac{x}{y} = \frac{c}{b}$.

$\Leftarrow$ Assume $\frac{x}{y} = \frac{c}{b}$ and M is the midpoint of $\overline{BC}$. Using the same area logic as above, $\frac{c}{b} = \frac{u}{t}$ and

Fig. 12



so, by The Isogonal Theorem, $\overline{AP}$ ($\overline{AD}$) and $\overline{AM}$ are isogonal. Since $\overline{AM}$ is a median, $\overline{AD}$ must be a symmedian.

13. Let $\overline{AD}$ be a symmedian and let $BD = m$ and $DC = n$. From problem 12 above, $\frac{x}{y} = \frac{c}{b}$.

Because they share the same altitude, $\frac{K_{ABD}}{K_{ADC}} = \frac{m}{n}$.

But $K_{ABD} = \frac{1}{2}xc$ and $K_{ADC} = \frac{1}{2}yb$. So

$$\frac{m}{n} = \frac{xc}{yb} = \left(\frac{x}{y}\right)\left(\frac{c}{b}\right) = \left(\frac{c}{b}\right)\left(\frac{c}{b}\right) = \frac{c^2}{b^2}.$$

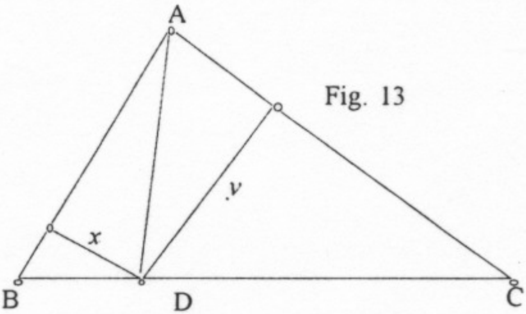


Fig. 13

14. Consider ΔABC with $\overline{DE}$, a parallel to side $\overline{BC}$ and $\overline{AX}$, the angle bisector (Figure 14a). Let $\overline{D'E'}$ be the reflection of $\overline{DE}$ through $\overline{AX}$. Using congruent triangles, it is easy to prove that $\Delta ADE \cong \Delta AD'E'$ and so $\angle ADE \cong \angle AD'E'$. Therefore, $\overline{D'E'}$ is an antiparallel to side $\overline{BC}$.

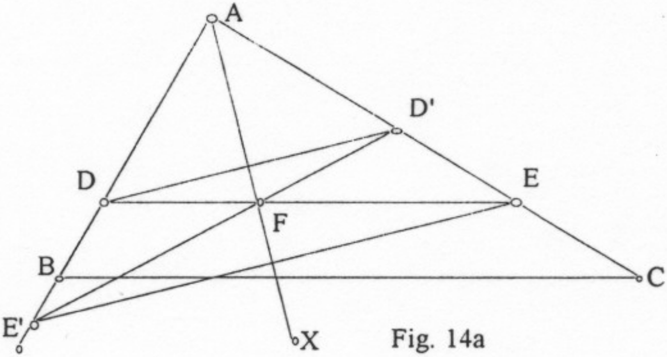


Fig. 14a

Using Figure 14b, since the median $\overline{AM}$ bisects $\overline{DE}$ at G and the fact that the reflection transformation preserves distances, its reflection through $\overline{AX}$ will bisect the reflection of $\overline{DE}$ through $\overline{AX}$. Therefore, the symmedian from A bisects any antiparallel to $\overline{BC}$.

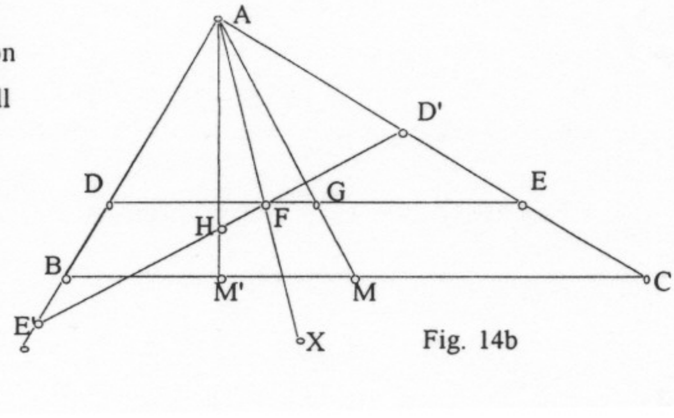


Fig. 14b