Attend the 29th Annual National Convention in Gatlinburg!!

THE MATHEMATICAL LOG

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# Counting Infinities

*by Richard Rusczyk*

Anyone comfortable with numbers knows that it's senseless to speak of counting to infinity, but it is feasible to compare two sets of infinite size. For example, how can we compare the set of positive integers to the set of all integers? It would seem that the set of positive integers must be smaller than the set of all integers because the set of positive integers is a subset of all integers. However... Before we proceed with counting infinities, let's chat a bit about our best tool for comparing infinite sets.

## 1 To 1

Whenever confronted with a set we don't know how to count, our best bet is to try to compare it to something we do know how to count. Even better is to find a one-to-one correspondence between what we want to count and what we know how to count. By one-to-one correspondence, we mean that each object in one group corresponds to exactly one object in the other. As an example, suppose you and your sister enter a CD club that lets you buy 20 CDs for one penny. You flip a coin to decide who pays the penny, then you each pick 10 CDs. You know you want to buy some rock, some country, and (trusting your sister won't tell your friends) some opera. While you're pondering the different combinations, you get curious about how many different ways you can split the 10 choices among the three genres of music (again trusting your sister not to tell your friends).

You think back to your algebra classes when you did all those balls in boxes problems, and try to think of this as putting 10 balls in 3 boxes, one for each genre. Well, you could put 10 in the first, or 10 in the second, or 10 in the third. That's 3. Let's check 9 balls in one box, one ball in another. That'll give us six more (9-1-0,9-0-1,0-9-1,0-1-9,1-9-0,1-0-9). Hurm, this casework seems like it will take an awfully long time. Let's try something else before coming back to this.

Suppose we line the CDs up – first rock, then country, then opera. We have eight slots and 2 dividing points between the slots.

Rock/Country/Opera
XXXXX/XX/XXX

This looks a little more promising. Anytime we write a series of 10 Xs, we can drop in two slashes like above and create a partition of the 10Xs into the three categories. Conversely, given a division, such as 7 rock, 3 country, and 0 opera, we can write 'XXXXXXX/XXX/' to represent that choice. We thus have a one-to-one correspondence between CD choices and 12 letter words made up of 10Xs and two /'s. While we have nothing better than lists to count the former, the latter we are familiar with. We can form the 12 letter word by choosing the two slots for the /'s. This we can do in

$$\binom{12}{2} = 66$$

ways, so we've solved our problem. We can order the Xs and /'s in 66 ways, so we can choose our CDs in 66 ways.

## And Now, Back to Our Story

Armed with the technique of one-to-one correspondences, we return to our problem of comparing the set of integers to the set of positive integers. Can we create a systematic one-to-one correspondence between the two sets? If so, then we must conclude that they are the same size. Well, let's try counting from 1 for the positive integers, and from 0 for all integers as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | -1 | 1 | -2 | 2 | -3 | 3 | -4 | 4 | -5 |

Looks pretty good up to 10. To generalize, we can map all odd positive integers, $n$, to $(n-1)/2$, and all even ones to $-n/2$. In reverse, we map each negative integer, $m$, to $-2m$ and positive ones to $2m+1$. Convince your self that this correspondence is in fact one-to-one. Since for each positive integer we can choose a distinct integer from all integers, and vice verse, these two sets are the same size. Sounds insane, no?

## Seems Rational

What about rational numbers, numbers which are expressed as the ratio of two integers? Since we can think of these as two dimensional numbers, numerator and denominator, let's draw them as such:

*...continued on page 5*

# News and Announcements

## •National Convention '99

The 1999 Mu Alpha Theta Convention will be in Gatlinburg, Tennessee. The Convention will run from August 1st to August 6th. For those who have never attended a national convention, it is an excellent opportunity, not just to learn mathematics from the speakers and the contests, but to meet other students and teachers from around the country. *Editor's note: I attended four conventions as a student, and throughout a high school math 'career' which covered nearly a hundred tournaments and three trips to the Math Olympiad Program, I didn't enjoy any other math event more than I did the national conventions.*

The contact person is Grace Mutz or Mary Emma Bunch at Farragut HS, 11237 Kingston Pike, Knoxville, TN 37922 and the phone number is (423) 671-7154. Packets will be mailed to those schools who have come to conventions the past few years. If you are new to our conventions, please write to either Grace or Mary Emma or the national office to receive a convention packet.

## •Increased Dues

The motion to increase annual dues from $3 to $5 has passed.

## •The Mathematical Log

Some readers may have noticed that you missed the December Mathematics Log. No need to call the Post Office and demand they deliver your copy. Mu Alpha Theta has decided to reduce publication of the Mathematical Log from 4 issues a year to 3 per year.

## • State and Regional Meetings

February 12-13, 1999 - Texas state meeting- San Antonio, TX. For further information contact Oscar Castaneda, Southwest HS, 11914 Dragon Ln., San Antonio, TX 78252

March 28-29, 1999 - Mississippi State Meeting, Jackson, MS. Contact Claudia Carter, MSMS, PO Box W-1627, Columbus, MS 39701.

## •MAΘ Website

For those of you who don't know yet, Mu Alpha Theta is now online. You can check out the website at www.mualphatheta.org for all the latest information from the national office.

## •National Convention Registration

Coming soon to a website near you: registration for the national convention the easy way. Check the Mu Alpha Theta website for details at the end of February!

## •Student Articles

If there are any budding mathematical writers out there among our student readers, please submit articles to the Mathematics Log for publication. Please send all articles to the National Office (address below).

The Mathematical Log
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# $\sqrt{\text{At the Root of it All}}$

## *Deborah Patonai Phillips, Activities Editor*

Powerful things often come in very small packages. This is definitely a true description of the 1998 Huneke Award winner Mary Emma Bunch from Farragut High School in Knoxville, Tennessee. A petite, soft-spoken dynamo of a mathematics teacher for over two decades, Mary Emma has impacted hundreds of students in the classroom as well as in Mu Alpha Theta and in the Math Team. A Mu Alpha Theta sponsor of tremendous vision and ability, she has given freely of her leisure time and of her own personal finances to further mathematics education. Harold Huneke, former MAΘ Secretary-Treasurer, would be proud to recognize her as an outstanding MAΘ sponsor.

In the classroom, Mary Emma sets the stage for learning. Because of her exceptional grasp of mathematics, her students perceive her to be the knowledgeable, enthusiastic, and helpful individual that she is; consequently, their response to her produces the desired effect—they learn. According to her principal, "She not only relates to her students in a manner which is conducive to productive use of time but also instills in them a desire to learn as well as a belief in their own ability. She creates and maintains a warm, open climate wherein there is purpose with flexibility and control without dominance."

In 1979 Mary Emma, along with her colleague, launched the first MAΘ chapter at Farragut. Enjoying the opportunity to take mathematics out of the classroom, she created a group of like-minded students and friends who share not only the lure of competition but also leisure moments.

Turning mathematics into a hobby for her students, Mary Emma instills in them confidence as well as a competitive spirit. When there is a task at hand, Mary Emma does not waste time with frivolous instructions; she just allows the club members to make key decisions. However, she is constantly monitoring to assure the success of her students and will, when necessary, step in and direct. Exemplifying special qualities that keep MAΘ on target, Mary Emma "is the glue which holds us together as well as the spark which ignites our competitive spirit," reports Aram Yang, former Farragut MAΘ president and national vice-president.

From the beginning of her involvement with MAΘ, Mary Emma has touched the lives of students beyond Farragut High School. She has attended every state convention with the exception of one and has attended sixteen of the last eighteen national conventions. In fact, selfless with her time, her energy, and her expertise, she has co-hosted numerous local, state, and regional mathematics competitions. She co-hosted the 1988 National MAΘ Convention at the University of Tennessee, Knoxville, and will co-host the 1999 National Convention this summer at Gatlinburg, Tennessee.

Besides the competition aspect of MAΘ, Mary Emma has been instrumental in helping other chapters get chartered and has even assisted with initiation ceremonies at other schools. Most recently, she has chaired the Student-Host Committee for the National Council of Teachers of Mathematics regional meeting. At the national level, Mary Emma has served as NCTM Representative on the Governing Board and is currently the chairperson of the Convention Committee.

Outside of MAΘ, Mary Emma has continued her professional growth by interacting with colleagues in her school as well as by serving as an active participant in numerous professional organizations. Often taking new teachers under her wing, she works tirelessly to instill high standards in education. Due to her efforts, Farragut's mathematics teachers are better versed and more aware of current trends in mathematics education. In addition to serving in leadership positions within her school, such as the Faculty Committee for Scholarships, she has been elected to three offices in the Smokey Mountains Mathematics Educators' Association, including president. Because of her varied efforts and her dedication to the improvement of mathematics education, she has been named Teacher of the Year by her faculty on four separate occasions, has been selected as a finalist for Best Teacher for Knox County and for the Presidential Award for Mathematics for the state, and has been named a Teaching Fellow by the University of Tennessee.

"Enthusiastic, energetic, wise, and caring, Mary Emma Bunch has made a decided impact in the mathematics program at our school and in mathematics awareness in our community," relates her co-sponsor Grace Mutz.

Her unparalleled success in the classroom, in MAΘ, and in her profession may be due partly to her demeanor, which could be best described as Southern friendliness. Her genuine concern for the individual is obvious in the manner in which she treats everyone and in her kindness in whatever group she is directing. "Students like her," explains one of her former students, "and because of this, they, too, go the extra mile in learning. . . She is an authority figure who is approachable, a teacher who is also a friend, and a personal cheering section for each member of our club."

Even though Mary Emma comes in a small package, she offers not only her sage advice but also her ideas, her expertise, and her willing assistance to her students and to her profession. Congratulations, Mary Emma, on a well-earned honor.

# First or Fun?

*By Ryan Goldenberg,*
*National MAΘ Secretary/Treasurer*

When I first joined Math Club in 8th grade, I had entirely different reasons for participating than I do now. I was less mature at the time and only thought of attending competitions in an attempt to win trophies and to place Berkeley Prep higher in the sweepstakes awards. As the years went on, however, I learned something new, other than how to be competitive.

Although I do think that the first reason for attending should be to compete, I feel other reasons for attending also apply to many activities in school and even in life. My prime motive for waking up as early as 5:30am is still my desire to be the highest scorer. However, now in my senior year, I look forward to many other important aspects of math competitions.

I feel that the second-most significant reason for attending competitions is the people that one has the opportunity to meet. And what's great, with today's technology staying in contact with people you meet just once does not cost money! True, long-distance phone calls can definitely put a large dent in the phone bill, but e-mail and other forms of online communication can create valuable friendships. Even if one never gets to see the other again, it is still comforting to have someone with common interests to talk to.

I have met several competitors at regional, state, and national conventions. Fortunately, I have had the opportunity to spend time with many of those who live in my area, and our mathematical relationships have become more like those of friends. For those who live out-of-state, I'll just say that my e-mail inbox is full when I come home from school everyday!

Another advantage of attending math conventions resides in the material and knowledge that one may gain. Obviously this information improves one's performance, but I have found that the topics I have learned may actually be used in the coming years for high school or college courses. Additionally, the tests themselves require one to think logically. With much practice, this process of reasoning will develop and may assist one in making decisions on life's long path. Although usually the best competitors with the most knowledge come out on top at the awards ceremony, everyone has an occasional unlucky day, or perhaps something may go wrong with the test itself. But these unfortunate events challenge students and force them to handle difficult situations. One must realize that everything does not always go smoothly, and that one must sometimes face disappointment. I have learned to cope with these "bad days," and am a better person simply because I do not allow myself to break down and fret about the misfortune.

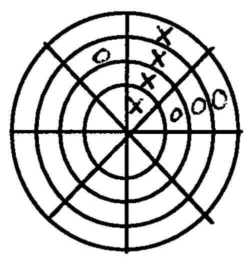
The last and best way to end a math convention is with memories of a fun event. Although I take the competitions seriously, when it is time to take a break from the tests and relax, I like to take advantage of the moment! Constantly studying and practicing will indeed prepare one well, but it can also get tiring. Taking a relaxing break is almost necessary, for it helps to remove any uneasy feelings about the tests, as well as to relieve stress.

This August at nationals, I will participate in my last competition as a student, and I hope to end on a good note! I want to be remembered not only as a strong competitor, but one who is also considered to benefit from all aspects of math conventions aforementioned. I have talked to Jennifer, Paul, and Vivek recently, and look forward to a lot of fun activities taking place during the week of nationals!

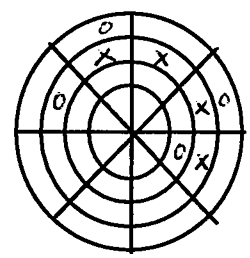
See you all in Gatlinburg!

# Games People Play

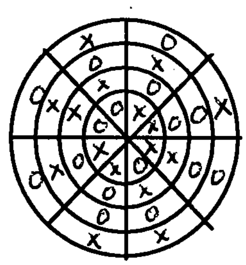
How about a game of Tic-Tac-Toe? Too easy, you say? Let's play it on a circle. Scribble four concentric circles, then divide them into eight sections with four diameters. Take turns with your x's and o's. The three winning moves are four in a row radially (straight out from the center), the semi-circle (four in a row in one of the four rings), or the whorl (four in a sprial going out from the center). No winning moves can go through the center. Here are three examples of winning formations and one board in which neither side wins.



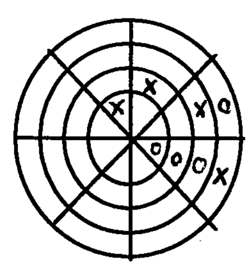
X wins



X wins



no one wins



X wins

## Counting Infinities

*...continued from page 1*

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Can we write positive integers in such a two dimensional format? Sure. Start from the upper left and fill the grid diagonally:

$$\begin{array}{ccccc} 1 & 2 & 4 & 7 & \cdots \\ 3 & 5 & 8 & 12 & \cdots \\ 6 & 9 & 13 & 18 & \cdots \\ 10 & 14 & 19 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

There's our correspondence. What does the fact that some of the fractions in our first grid are reducible do to our proof that the set of rationals is the same size as the set of positive integers? Is it an irreparable hole?

### Get Real

What about all real numbers? Are these countable, meaning the same size as the set of integers? A question that puzzled mathematicians for quite a while before Georg Cantor came up with his elegant diagonal proof in the late 1800s.

If the reals are countable, we should be able to list them, one next to each integer it corresponds with in the one-to-one correspondence between reals and integers:

$$\begin{array}{ll} 1 & x_{10}.x_{11}x_{12}x_{13}x_{14}\cdots \\ 2 & x_{20}.x_{21}x_{22}x_{23}x_{24}\cdots \\ 3 & x_{30}.x_{31}x_{32}x_{33}x_{34}\cdots \\ 4 & x_{40}.x_{41}x_{42}x_{43}x_{44}\cdots \\ \vdots & \end{array}$$

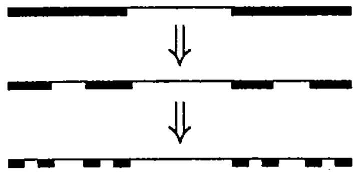
Now we build a new number as follows: let $b_1$ be any digit different from the digit $x_{11}$, $b_2$ be any digit different from $x_{22}$, $b_3$ be any digit different from $x_{33}$, and so on. The real number

$$0.b_1\, b_2\, b_3\, b_4 \cdots$$

is therefore different from each real number in our original list (because the $n^{th}$ digit in our new number is different from the $n^{th}$ digit in the $n^{th}$ number in our list for all values of $n$). So there are some real numbers missing from our correspondence, yet we've used all the positive integers. Therefore, there's no one-to-one correspondence between the reals and the integers. The reals, then, we call uncountably infinite.

### But Wait, There's More

Take a line segment of length one. Cut out the middle third. Next, cut out the middle third of the two remaining pieces, then the middle third of the subsequent four pieces. Continue cutting out middle thirds forever, as shown below.



The remaining set of points is called the Cantor set, after the aforementioned Georg Cantor, who must have had a great time playing with infinite sets. We started with a segment of length one. What is the length of the remaining set of points as we continue this cutting process indefinitely? At each step, we reduce the length by 1/3. After one step, it is 2/3, after two it is $(2/3)^2$, etc.; therefore, the length is tending toward zero as we continue snipping out middles. But this doesn't mean there are no points!

Consider the line segment to be a number line from 0 to 1 of all reals written in base three. The first cut removes all numbers whose representation starts with the digit 1. The second set of cuts removes those with first digit 0 or 2 but second digit one. The next snips nail those numbers with third digit 1, and so on. Therefore, the remaining points are all those whose base-3 decimal representation has no 1's. Cool. Now how many of these points are there?

All our numbers in decimal form contain just 0's and 2's, and in all possible combinations. Hmm. Just 0's and 2's. Just yesses and nos. What does that sound like? Yep, binary numbers. If we replace all our 2's with 1's and consider the resulting expression as a decimal in base 2, we get all possible decimals written in base two. This process is clearly reversible (take a number between 0 and 1, write in binary, change the 1's to 2's, value in base 3); hence, the Cantor set has a one-to-one correspondence with the real numbers from 0 to 1. So the Cantor set, despite having length 0, has as many points as the line segment it is formed from.

A couple little (tough!) exercises:

1) Show that the set of real numbers in any finite range, say from 0 to 1, is the same size as the set of all real numbers.
2) Show that the set of points in a square is the same size as the set of points in a line segment.

*Editor's note: Most of Counting Infinities is lifted from the Art of Problem Solving, by the Editor and Sandor Lehoczky. The Art of Problem Solving is available from Mu Alpha Theta. Contact the central office for details.*

## 8/2 * (8-2)

By popular demand, here are the answers to last issue's puzzles. For those who missed it, the challenge was to make 24 out of different sets of four numbers using only addition, subtraction, division, and multiplication.

| | |
|---|---|
| 1+7+7+9 = 24 | 4*5+6-2 = 24 |
| 2+2+4*5 = 24 | 3*3 + 3*5 = 24 |
| 3*9-5+2 = 24 | 3*8*8/8 = 24 |
| 9*(3-1)+6 = 24 | 2*(3*3+3) = 24 |
| 5+5+7+7 = 24 | 2*4*(8-5) = 24 |
| 6/(1-3/4) = 24 | 5*(5-1/5) = 24 |
| 4/(1-5/6) = 24 | 6/(1-6/8) = 24 |

### What Do You Want?

I'm open to suggestions for the Mathematical Log. What would you like to see? More interesting mathematics? History of mathematics? Applications? Articles on education? Philosophy of science and mathematics? Let me know what you'd like to see and you'll likely get more of it! Send your comments to:

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## BOOK REVIEW:

### *How to Solve It* by G. Polya

*By Richard Rusczyk*

Polya's *How to Solve It* is a classic, first published in 1945, among problem solving literature. Polya doesn't teach many specific methods to solve specific problems, but rather he focuses on the process by which most challenging problems can be solved.

Polya breaks the problem-solving process into four steps: understanding the problem, devising a plan, carrying out the plan, and checking the solution. These steps are pretty common to most teachers and students today, though most textbooks and many sources of education tend to gloss over the most important and most time consuming step, devising the plan, and focus instead on the other three. Polya instead spends much more of his writing on developing approaches to problems than he does on the other three steps.

Understanding the problem is both a duty of the problem poser and the problem solver. The question to be answered must be well specified along with the conditions and data necessary to lead to the solution.

Polya really shines in his discussions on building a plan. In fact, he treats the execution of the plan as large a trivial antecedent of an effectively developed plan. He walks through several approaches, including induction, proof by contradiction, reductio ad absurdum, working backwards, and solving constructions by relaxing conditions. Rather than just presenting these methods cookbook-style, Polya starts from examples and develops the methods while attacking the problems by asking a series of questions, often prompted by a little experimentation. In this way, he closely mimics the thought pattern of an experienced problem solver. He doesn't jump straight to the answer as most texts do, but spends time on investigation. This experience is the true value of the book.

Upon solution of many of the examples he introduces the importance and methods of checking the solution. Checking dimensions, making sure the answer makes sense, and assuring that variables which are symmetric in the problem are also symmetric in the solution are frequently used examinations of the solution. Furthermore, he encourages the solver, once the solution is found, to check if the answer suggests alternative approaches.

The primary drawback of *How to Solve It* is a matter of style. Polya presents his problem solving suggestions as a dictionary. He perhaps intends it to be used as a reference book. Unfortunately, his dictionary has such non-intuitive entries as "Auxiliary Problem", "Can You Use the Result", "The Future Mathematician", and these entries are listed alphabetically rather than by subject or some other cognitively rational grouping. As a result, it is nearly impossible to use the book as the reference book it is laid out to be. However, the book shouldn't be used in the way an encyclopedia or a dictionary would be, so the fact that it can't be used that way is no great loss. That said, the book would have been a little more approachable and more interesting to read if it had a flow dictated by more than alphabetical order.

*How to Solve It* should be a staple in the libraries of teachers who are training students in advanced problem solving. While the thought processes Polya presents are certainly valid for approaching problems of all kinds and all levels, the book is really most useful for those attacking complicated problems. He provides several examples as well as suggestions on how to approach them with students. Would-be problem solvers will find in *How to Solve It* the answer to the question, "How would I ever think to do that?" Experienced problem solvers will find a clear discussion of the thought processes they follow by instinct.