Attend the 28th Annual National Convention in Chicago!!

THE MATHEMATICAL LOG

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You Wanna Bet!

Probability and Number Theory in Games of 'Chance'

Let's play a little game. I'll put one dollar on the ground, then I'll flip a coin. If it comes up tails, you get the money; if it comes up heads, I'll double the amount of money on the ground and flip again with these same rules. Thus, for each heads I flip before getting a tails, your payout from the game doubles. Sounds like a good deal for you, but I'm not going to let you play for free. You have to pay me to play. The question is, how much are you willing to pay?

You'll clearly be willing to pay at least one dollar, since you can't win any less. I, of course, won't let you play for that little. There's a 50% chance you'll win only one dollar, and a 50% chance you'll win at least two dollars, so you should be willing to pay at least 0.5(1) + 0.5(2) = 1.5 dollars. Continuing this logic, if we look three flips ahead, there's a 50% chance of one dollar, 25% chance of getting two dollars, and a 25% chance of getting at least four dollars. Hence, you should be willing to pay

$$0.5(1) + 0.25(2) + 0.25(4) = 0.5 + 0.5 + 1 = 2$$

dollars. Hmm. This is starting to look a little suspicious. If we consider n flips, your expected income is at least

$$\frac{1}{2}(1)+\frac{1}{4}(2)+\frac{1}{8}(4)+\cdots+\left(\frac{1}{2}\right)^{n-1}(2)^{n-1}$$
$$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}+1=\frac{1}{2}(n-1)+1.$$

Interesting; since there is no limit on how many flips the game will last, there is no limit on what you expect to win. So theoretically, you should be willing to pay any amount of money to play this game with me, as any finite amount of money you pay is less than the expected value of the game. So now we're back to our original question: how much will you pay to play? Given our analysis above, you might easily be persuaded to play for twenty dollars. There's a slightly less than 1% chance you'll get over \$100, nearly a 0.1% chance you'll get at least \$1000, and about a two in a million chance you'll clear one million dollars! Much better odds than a lottery ticket. Surely you could be convinced to play for \$10 (I've witnessed people pay \$30 to play). Is that a fair price? If we look closely at our payout profile which we used to calculate the 'value' of this game, we see a bunch (in fact infinitely many) of scenarios that involve your winning well over one million, or one trillion, or even one quadrillion dollars. Think I can pay you that much? Not likely. So each of those scenarios, which in our calculations above contributed a paltry 0.5 towards our expected value, in fact contribute something much closer to 0.

Let's suppose I can only afford to pay you \$5000, and if you win any higher amount than that, I'll pay you only \$5000. For any number of heads up to 12 heads, I'll be able to make the regular payout as before, but if there are more than 12 heads, I'll only give you \$5000. Hence, your true expected value is:

$$=\frac{1}{2}(1)+\frac{1}{4}(2)+\cdots+\left(\frac{1}{2}\right)^{13}(4096)+5000\left(\left(\frac{1}{2}\right)^{14}+\left(\frac{1}{2}\right)^{15}+\cdots\right)=$$
$$13*\left(\frac{1}{2}\right)+5000*\left(\frac{1}{2}\right)^{13}\approx 7.$$

If you think I'll pay you no more than \$5000, you're paying too much if you give me \$10 to play.

A few follow-up questions:

1) How much would you be willing to pay Bill Gates to play this game?

2) What would the value of this game be if we used a lower multiplier than 2 for each head (but more than 1, of course)? What if we required that 2 tails come up before I finally pay you?

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News and Announcements

•Elections

In February, Mu Alpha Theta will be conducting elections for the offices of President-elect, Secretary-Treasurer, Governor of Region III and Governor of Region IV. The position of Secretary-Treasurer is typically a professor affiliated with the University of Oklahoma (so as to be at the national office). This post has been very ably handled for the past four years by Dr. Stanley Eliason, and nominations for his continuing will be accepted.

If you have any nominations, please send them to Chair of the Nominating Committee:

Barbara Stott
Riverdale High School
240 Riverdale Dr.
Jefferson, LA 70121.

•Regional and National Meetings

Region IV Meeting
December 5-6, 1997
Carter High School
Strawberry Plains, TN
for further information email Cardinaldix@hotmail.com

•If I Were in Charge...

What do you like about national conventions? If your chapter doesn't participate in the national convention, what changes or additions would bring your chapter to the conventions?

The Conventions Committee, created by the Governing Board, is charged with revising and rewriting the "Guidelines for Convention Hosting," written in 1990. Now in its second year the committee is hoping to standardize certain portions of each national convention (i.e., opening ceremonies, number and types of tests, length of conventions, chalk talks, speaker sessions, sweepstakes, poster and banner competition and a general program of events). In doing so, this will offer the convention host the freedom to include 'local' events as well. The committee will also look at the schedules of pending conventions and make recommendations to the host school(s) on improvements.

If any MAΘ chapters, sponsors, or members have any convention suggestions please send them to the committee chair:

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•Math Contest on the Web

This year The Mandelbrot Competition, created by Mu Alpha Theta members Sandor Lehoczky and Richard Rusczyk along with Samuel Vandervelde kicks off its eighth year by starting a web-based contest in which tests can be downloaded from the web and test scores can be reported directly. Therefore, results can be compiled more quickly and information distributed and shared among participants more easily. The contest also inaugurates a middle school contest, Mandelbrot Midlevels, and adds five new question writers. For more information on the contest, visit their website at www.mandelbrot.org.

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$\sqrt{\text{At the Root of it All}}$

Deborah Patonai Phillips, Activities Editor

Wandering around this year's national MAΘ Convention in Seattle was a new face, that of recently elected MAΘ President Dr. John A. Dossey. Briefly at last year's convention in Orlando, John witnessed that MAΘ is really great for students who like mathematics and who enjoy using mathematics to solve problems. However, this was not John's first exposure to MAΘ; thirty-five years ago he was a MAΘ member at Peoria High School in Illinois. Remembering that some of his closest friends today were in his MAΘ chapter, John can attest to the fact that "friends developed through MAΘ may last a lifetime." He also thanks MAΘ for introducing him to a career in mathematics.

$\sqrt{\textit{At the Root of it All}}$ welcomes back one of MAΘ 's own. Holding B.S. and M.S. degrees in Mathematics from Illinois State University and a Ph.D. in Mathematics Education from the University of Illinois at Urbana, John has taught at the junior high, the senior high, and the university levels. After teaching grades 7 through 12 in the public schools of Normal, Illinois, he then served as its K-12 Mathematics Supervisor. In 1967 he joined the faculty at Illinois State, where he was responsible for the secondary mathematics education program.

Teaching not only mathematics education courses but also mathematics courses, John has influenced future teachers from undergraduate to doctoral levels. Receiving special distinction during the 1993-94 school year, he acted as the Distinguished Visiting Professor of Mathematical Sciences at the U. S. Military Academy at West Point.

John brings to MAΘ a wealth of experience. He has benefited the mathematics community in a number of important roles. As President of NCTM, he led them through the development of their landmark Curriculum and Evaluation Standards for School Mathematics. As Chair of the Mathematical Sciences Advisory Board for the College Board, he was the lead developer for the College Board's SAT II testing program in mathematics and a reviewer for other assessments in mathematics developed by the Board.

Since 1986 John has been involved as one of the principal authors of the National Assessment of Educational Progress (NAEP) reports in mathematics. As part of this role, he is a member of the test development committee which works to establish scoring standards for the open-ended items and the methods of reporting the data at the state level. In addition to these commitments, he serves the National Research Council as Chair of the U.S. National Commission on Mathematics Education, as a member of the Board of International Comparative Studies in Education, and as a participant on the U.S. Steering Committee for the Third International Mathematics and Science Study (TIMSS).

At present, he is Chairman of the Mathematics Committee charged by the Secretary of Education with the task of designing and of developing specifications for the National Voluntary Mathematics Examination proposed by President Clinton. Well-known for his development of curricular materials for students in grades 7 - 12, John has authored or co-authored more than 60 books and 150 chapters, papers, and associated publications. He is co-author of both the Scott-Foresman/Addison Wesley Middle School Mathematics Program and the Addison Wesley Secondary Mathematics Program. At the college level, he is the co-author of textbooks in discrete mathematics as well as content and methods textbooks for elementary, middle school, and secondary school teachers of mathematics.

John has earned numerous awards and honors as a result of his involvement with mathematics. Besides his Distinguished University Professor Chair at Illinois State, he has received the Lifetime Achievement Award from NCTM, the Max Beberman Award for Research from the Illinois Council of Teachers of Mathematics, and the James Armstrong Award for Teaching from the Illinois Mathematical Association of Community Colleges. John has also been granted the Outstanding Civilian Service Medal from the Department of Defense and an honorary Doctor of Human Letters degree from Eureka College.

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And now for a word from...Zubin Teja, Student Delegate President

Hello! I'm Zubin Teja, your Student Delegate National President. I am currently a senior at J.P. Taravella High School in South Florida. I have been a Mu Alpha Theta member since my freshman year and have attended numerous Mu Alpha Theta conventions and competitions. Besides Mu Alpha Theta, some of the activities I am involved in include debate and forensics, computer programming, and cross-country running. I hope everyone is enjoying their return to school.

This summer I joined about a dozen students from my school for an exciting week of fun at the National Mu Alpha Theta Convention in Seattle. The convention featured a scavenger hunt, Mu Alpha Theta-style Jeopardy, several dances, perspicacious guest speakers, and many challenging competition tests including relays, bowls, ciphering, and various topic tests. We also visited Pike Place Market, took a boat ride to Tillicum Village, went on a tour of Microsoft, spent a day in downtown Seattle, and traveled up the world-famous Space Needle. The convention ended with a formal banquet, at which awards were given out and everyone recalled the great times they had enjoyed over the past week. Many new friendships were made while old ones were renewed, and the friendly competition stimulated the minds of all. Many thanks to Tom Norris and the Thomas Jefferson students for being incredible hosts.

Next year's convention will be held in Chicago, and promises to be the best one yet. I encourage those chapters from around the nation which have never attended a national convention to join us in Chicago, whether you bring one student or 50 students. Just contact the national office or the Chicago convention committee for a convention packet and more details. I assure you that once you attend, you'll keep coming back.

For those of you interested in fund raising to subsidize the costs of attending the Nationals, here are a few techniques we have implemented in my school's Mu Alpha Theta chapter. First, consider hosting a local or regional competition. Not only will it give you a behind-the-scenes look at how a competition is run, but, by charging a small fee, you can generate quite a bit of money. Second, seek out the financial support of companies and organizations in your community through corporate donations. Many would happily support a group of students attending a prestigious and educational convention such as ours.

The other officers and I are actively pursuing the suggestions and concerns brought up at the student delegate meeting held at the Seattle convention. If you have any questions, concerns, or comments, or if you would just like to share ideas, please don't hesitate to contact me, Aram, Chris, or Nathan at any time. We are here to represent you!

Good luck to everyone and have a great year! See you in Chicago.

√*At the Root of it All*

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Even though John has traveled the world in the name of mathematics, he was still very impressed with the MAΘ convention in Seattle. Thinking that the competitions, the speakers, the activities, the trips, and the outings were "tremendous," he noted that "everyone involved had a great time and came away richer for having visited the Northwest and participated in the MAΘ convention." In his role as president of MAΘ, he hopes to make future conventions even bigger and better by bringing the Governing Board and the Student Officers into closer contact. MAΘ is indeed fortunate to have John Dossey as its president. Drawing from his vast resources of contact in the mathematical community, he will help to make MAΘ a stronger and a more developed organization.

Welcome back to MAΘ, John!

You Wanna Bet!

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Pick-up Sticks

I'm going to put 23 sticks on the ground, then we're going to play a game in which each of us take turns removing 1, 2, or 3 sticks and the person who is forced to take the last stick loses. Since this is my game, I get to go first. I choose two sticks and proudly proclaim that I have won. Undaunted, you point out that the rules clearly state that you don't lose until you are forced to take the last stick, not the 21st. OK, if you insist on playing, I say.

You choose 3 sticks, I choose 1, leaving 17. You choose 2, I choose 2, leaving 13. You pick 3, I pick 1, then you pick 1 and I pick 3. There are only 5 sticks left. And it's your turn. Suddenly you don't like your chances. No matter what you pick, I can leave you with the last one. You pick 2, I pick 2 and you lose. In order, then, the number of sticks remaining at each point in our game is

$$23,\underline{21},18,\underline{17},15,\underline{13},10,\underline{9},8,\underline{5},3,\underline{1},$$

where the underlined numbers show when you are picking. We play again and get the sequence

$$23,\underline{21},19,\underline{17},18,\underline{13},10,\underline{9},6,\underline{5},2,\underline{1}.$$

You lose again. One more time:

$$23,\underline{21},18,\underline{17},16,\underline{13},12,\underline{9},7,\underline{5},4,\underline{1}.$$

Again you lose. After scribbling out these sequences, you see that you're always choosing from the same set of numbers. Why do I make you do this?

If you are choosing from 5, you'll clearly lose. Once you pick n sticks, I'll nab $4-n$ and you're left with 1. If you choose from 9 sticks, I can force you to choose from 5 sticks in just the same way. Continuing backwards, you lose if you choose from 13, 17, or 21 sticks. Whoever goes first should win.

Follow up questions for pick-up sticks:

3) How should you play if there are k sticks and each person can choose from 1 to j sticks?

4) How should you play if the object is to choose the last stick?

5) How does the game change if one player can choose up to i sticks and the other can choose up to j, where $i>j$?

More coins

You're a little wary, or perhaps weary, of playing games with me, but I'm going to trouble you to play one more. You tell me a sequence of three coin flips and then I pick a sequence and tell you. We're going to flip a coin repeatedly until one of our sequences comes up as three consecutive flips. Since you're a little skeptical about playing with me, I'm going to be a sport and give you good odds. If your sequence comes up first, I'll give you \$3; if mine wins, then you give me \$2.

Sounds good; it seems that any two sequences are equally likely to come up first, so you choose what comes to mind first, HHH. Wearing my best poker face, I pick THH. We start flipping. First toss is a head. Next one is a tail. Pay up, I tell you. Not again, you think. Before objecting, you realize that once a tail has been tossed, you can't win. Since the very first sequence of HHH must be preceded by a tails (unless the first three flips are all heads), my THH will always come one flip before your HHH. The only way you can win is to get three heads right away. You therefore have a 1/8 chance to win if you pick HHH.

What if you pick something a little more exotic, like THT? How can we analyze that? Well, for each flip, all that matters is what happened on the prior two flips. Thus, we only have to analyze the four cases HH, HT, TH, TT. After we flip a coin, if neither of us win the game, then we've just moved to another of the four cases. For example, if the first two flips are heads, so that we're in the case HH, another head will leave us at HH while a tails will move us to HT (since the last two flips were a heads, then a tails, in that order).

The linkings among these four cases are shown in the diagram on the following page.

We have nodes for each of our two-flip cases and arrows out of each of these nodes to represent what node we move to upon flipping the coin again. To check out our HH example from above, look at the HH node. The arrow (1) shows what happens if a head is next; we stay at the HH node. If a tails comes up next, we go to the HT node. There are 8 arrows, and each corresponds to a three-sequence (arrow (1) is HHH; arrow (2) is HHT).

This picture quickly reveals the folly of choosing HHH against my THH. Arrow (1) is your winner, while arrow (8) is mine. The only way you can win is to be on the HH node and get another heads. The only ways to get

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to the HH node are to either start there (a ¼ probability, as all four nodes are equally likely to be the starting point), or to get there via arrow (8), in which case I win. Hence, as before, the probability that HHH beats THH is the ¼ chance that HH is the starting node times the ½ probability that the next flip is heads, or 1/8. Once any other node is reached, HHH is sunk.

So let's compare THT to a few of the other possibilities. First, I clearly don't want to pick HHH or TTT; these are dominated by THH and HTT, respectively. Suppose I take HTH, so that you have arrow (7) and I have (4).

Starting from either node TH or node TT is equivalent because TT will eventually lead us to node TH along arrow (6) and neither of us can win before that happens (because neither of us has arrow (6) or arrow (5)). Similarly, nodes HT and HH are equivalent. By symmetry, the probability that I win from nodes HH and HT are the same as yours from TT and TH. Therefore, we are equally likely to win.

That's no good. I want an advantage. Perhaps I can get an advantage by 'stealing' one of your nodes. Suppose I pick TTH. Now you can't get to node TH along arrow (6), because I'll win. In fact, I always win if we reach node TT. From node HT, there's a ½ chance we'll take arrow (3) to node TT and I'll win, and a ½ chance we'll go to node TH. Once at node TH, there's a ½ chance you'll win (arrow (7)) and ½ we'll return to HT via node HH. Hence starting from HT, there's a ¼ chance we'll return to HT before someone wins and:

P(I win from HT) = ½ + ¼*P(I win from HT).

Thus, my probability of winning from HT is 2/3. As before, node HH is essentially the same as HT since we must go from HH to HT before someone wins. Starting from node TH, there's a ½ chance you'll win right away and a ½ chance we'll go to HT through HH. Thus,

P(I win from TH) = ½*P(I win from HT) = 1/3.

Since each node is equally likely to be our starting point, the probability I win is

¼(1+2/3+2/3+1/3) = 2/3.

Don't you hate playing games with me?

Follow up questions:

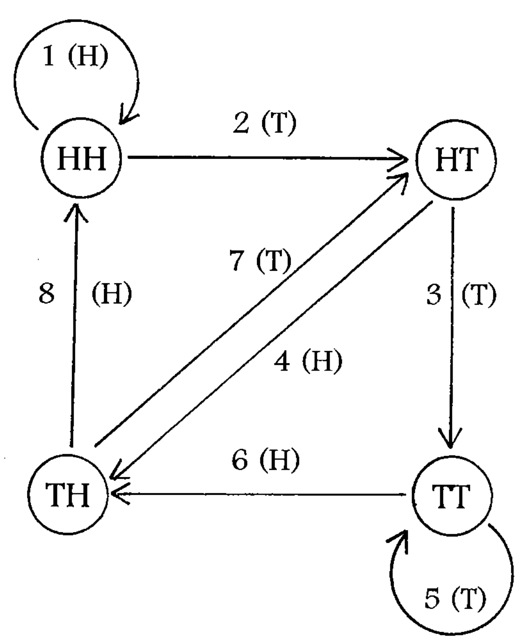
6) What should I choose if you pick TTH? THH?

7) How do your odds change if we use 4 coins? 5 coins? *n* coins?

For the physics buffs among you (saving the coolest tricks for last):

8) Take a penny and spin it on a smooth surface 20 times. How many times does heads come up? Why?

9) Better yet, stand 20 pennies up on their edges on a table and gradually tap the bottom of the table until they all fall. How many are heads? Why?



What Do You Want?

I'm open to suggestions for the Mathematical Log. What would you like to see? More interesting mathematics? History of mathematics? Applications? Articles on education? Philosophy of science and mathematics? Let me know what you'd like to see and you'll likely get more of it! Send your comments to:

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