The Mathematical Log

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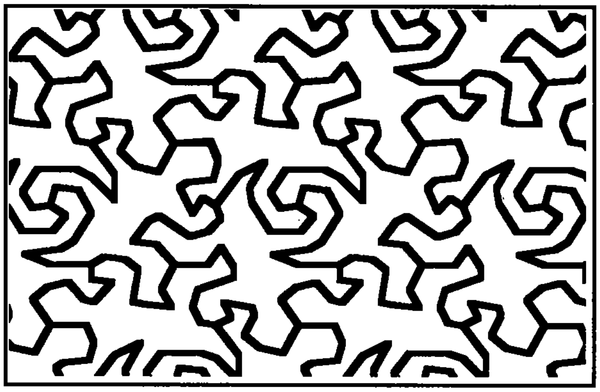
Tiling the Plane with Polygons



The problem of tessellating or tiling the plane was first studied by Johann Kepler in 1619. In its simplest form, the problem is to determine which polygons cover the plane without gaps or overlaps. Such tilings have been used in applications involving metallurgy, geology, biology, communications, architecture, and engineering. Many of you have probably seen a version of the tiling [above] by the Dutch artist M. C. Escher [1898 - 1972].

A **tiling** of the plane is a collection of closed regions called **tiles** such that

- no two of the tiles have any interior points in common
- the collection of tiles completely covers the plane

In this introduction, we will insist that the tiles are polygons joined **edge-to-edge**, each edge of a polygon in the tiling must be an edge of a one of its neighbors. A **regular** tiling uses only one type of regular polygon [Fig. 1] , a **semiregular** tiling has two or more regular polygons arranged the same way at every vertex. [Fig. 2]

Kepler found all regular tilings; we investigate the problem of finding all regular and semiregular tilings. [As you read on, stop from time to time and see if you can finish the solution for yourself before reading it.] We need at least three polygons to meet at each vertex. [What is the maximum number ?] The size [in degrees] of each interior angle of a regular n-gon is

$$\frac{180n - 360}{n} = 180 - \frac{360}{n}.$$

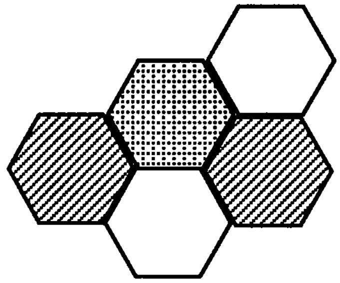


Figure 1

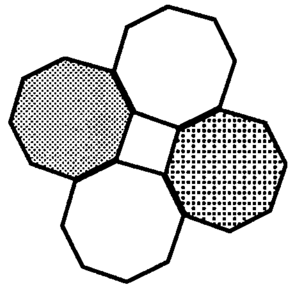


Figure 2

If a regular p-gon, a regular q-gon, and a regular r-gon meet at each vertex in a tiling, then

$$(180 - \frac{360}{p}) + (180 - \frac{360}{q}) + (180 - \frac{360}{r}) = 360.$$

This simplifies to $\frac{360}{p} + \frac{360}{q} + \frac{360}{r} = 180$ or

$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ – a diophantine equation [an equation whose unknowns are positive integers].

To systematically determine the solutions to this equation, we can asssume, because of the symmetry of the variables, that $p \le q \le r$. The smallest possible value of p is 3. Checking the possible values for q and r we find

p	q	r
3	7	42
3	8	24
3	9	18
3	10	15
3	12	12
4	5	20
4	6	12
4	8	8
5	5	10
6	6	6

Some of these are easily drawn; others appear not to work. To see why, suppose p is an odd number. Look at one of the p-gons in the tiling. As we go around it, we meet, alternately q-gons and r-gons, until there is a junction where things go wrong. [see figure on page 6]

[continued on page 6]

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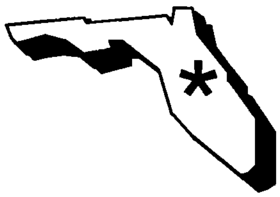
with Log Editor Tom Butts



A few final questions and answers from Sam Koski about the 26th Annual Convention at the University of Central Florida in Orlando, FL. Miami Killian SR HS, Miami Springs SR HS, and Miami Sunset SR HS are the hosts.

Question: Will it be possible for parents and/or other guests to attend the awards ceremony on Sun. Aug.11?

Answer: Good question! We foresee no problem with arranging for additional banquet space for guests. Additional information we be provided concerning cost and availability at a later date.

Question: What provisions are being made for housing boys and girls?

Answer: We have arranged for 11 floors of rooms at the University dorms. Each room has a sink and each pair of rooms shares a shower and toilet. We plan to have have each FLOOR be single sexed. If possible we would like to keep sponsors as close as possible to their students. Final arrangements can not be made until we have the completed roster from each school.

Question: At this point, how many openings remain for registration?

Answer: At this point in time (Saturday March 2 1996 there is plenty of space available. What is going to happen in the future is anybody's guess.

For more information write to 1996 National MAΘ Convention, P.O.Box 161166, Miami FL, 33116-1166; send an e-mail message to Sam Koski 76252.1660 @compuserve.com; or contact the national office.

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Logging On

Mu Alpha Theta has recently begun a web page [see √ At the Root Of It All]. Two internet sites with more information on tiling are:

- Math Forum: http://forum.swarthmore.edu/
This forum has a problem of the week including at least one past problem about tiling.

- Geometry Center: http://www.geom.umn.edu
This center has a QuasiTiler application for investigating tilings.

Students and sponsors: Send me information about your favorite mathematics Internet sites [and a better title for this new section].

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Unexpected Equations and Integer Solutions

Carl Michael Kelso of Jim Hill HS in Jackson, MS submitted a solution to theorem suggested by James Metz of the Mid-Pacific Institute in Honolulu, HI:

If (x_1,y_1) and (x_2,y_2) are the endpoints of the diameter of a circle, then an equation of the circle is
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

His solution involves using the circle formula
$(x - h)^2 + (y - k)^2 = r^2$ and noting that $h = \frac{x_1 + x_2}{2}$;
$k = \frac{y_1 + y_2}{2}$; and $r = \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{2}$.

Mr. Metz suggested a coordinate proof based on the Intersecting Chords Theorem. See if you can prove the theorem using this method. "What is beautiful about this story", Mr. Metz writes, "is not that a simpler method for solving a problem was discovered by a student, but that teachers can learn from students by listening to their suggestions. This has been my most rewarding teaching experience – and I was the one learning!"

Another of Mr. Metz's problems for you to think about: If a, b, and c are integers, under what condition(s) are the solutions to the equation $|ax + b| = c$ also integers? [Can the solutions be integers if a, b, and c are not integers?]
Send me your solutions.

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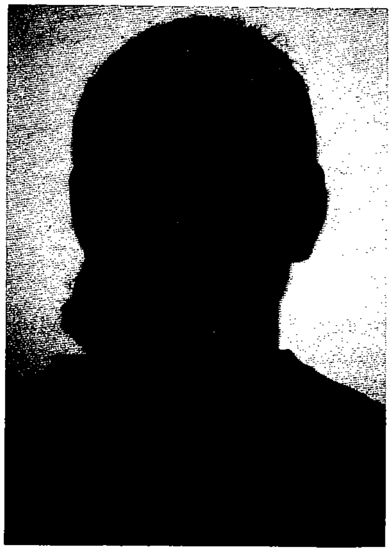
1996 √At the Root of It All

Deborah Patonai Phillips, Activities Editor
St. Vincent-St. Mary HS, 15 North Maple Street, Akron, OH 44303

Is there life after Mu Alpha Theta? Do the thrill of competitions, the stimulation of camaraderie, and the fascination of mathematics enhance the careers of former members? In at least three cases, the answer to all of these questions is a resounding "Yes". Aaron Lauve, Sandor Lehoczky, and Richard Rusczyk have all reappeared on the Mu Alpha Theta scene and rendered valuable service once again.



In Room 423 at 601 Elm Avenue in Norman, OK, otherwise known as the National Office, Stan Eliason and Diane Rubin coordinate the day-to-day operations of Mu Alpha Theta. A new addition to this team is former MAΘ member, Aaron Lauve, a "temporary" student employee whose main job has been to help the National Office join the computer revolution.

Born in Lafayette, LA, not too far from Diane Rubin, Aaron grew up in Mississippi. He found his way to Oklahoma through the Oklahoma National Scholars Program. After some dabbling in chemistry and engineering, he has decided to major in mathematics and physics before pursuing advanced studies or a job involving fluid dynamics. "In case you're wondering, this is a wide-open field with job opportunities ranging from chemical engineering to weather forecasting."

Mu Alpha Theta Web page:
http://www.math.uoknor.edu/~mualpha/

Aaron had to learn the powerful Paradox programming language in order to write programs to put nearly every function of the national office on computer – processing orders, inventory tracking, and database management. Recently he has been constructing and maintaining the Mu Alpha Theta Web page:

http://www.math.uoknor.edu/~mualpha/.
"The current version is in its infancy," explains Aaron, "but you might visit sometime. Comments and suggestions are welcome." This conversion will soon be complete. Many new services will become available to chapters. Need a list of chapters in your state? an official list of registered students in your chapter? Both are just a mouse click away.

Aaron offers some advice for current members. "It's not enough to pick a major when applying to college. I like chemistry and problem solving, so I thought chemical engineering was for me. .. Now I'm a math major. You must find out as much as you can." He challenges, "be studying some topology - you may find me hosting one of the workshops at this year's national convention."

As Aaron works to keep MAΘ in tune with technology, three other former members were putting on a show at the Texas State Convention. Sandor Lehoczky and Richard Rusczyk, both familiar names to Mu Alpha Theta with nearly ten national conventions under their belts as students or helpers, and Samuel Vandervelde hosted an "Evening at the Im π rov", a 90-minute show of improvisational routines based loosely [sometimes very loosely] on mathematical topics. Teachers and students joined in many of the skits. The most successful skit involved a student attempting to teach geometry to a class composed of teachers.

These three have collaborated on many other activities. Sam and Richard met at the Math Olympiad Program in 1987 while Sandor and Richard were fierce competitors in Alabama from 1987 to 1989. In the summer of 1990 while working on test for the 1991 Mu Alpha Theta convention in Huntsville, Richard came up with the idea of developing his own competition with his natural partners Sandor and Sam. The result was the *Mandelbrot Competition* now in its sixth year. While writing the contest and analyzing the feedback, Sandor noted the lack of comprehensive resources for students involved in "extracurricular mathematics". This led to the four-volume set, *Art of Problem Solving*, written by Sandor and Richard. Because of the books, a talk at MATHCOUNTS, and a year of teaching high school mathematics, Richard worked on *Sports Figuress*, part of the ESPN program *Cable in the Classroom*. . [see the last two issues of the Log for more information.]

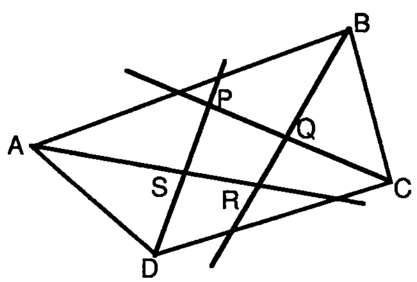
[continued on page 4]

Two Contest Books

There are many books of contest problems. Two new ones are now available from MathPro Press, P.O.Box 2039, Mansfield, OH 44905, e-mail: stan@MathPro.com, or web:http://www.mathpro.com/math/Mathpro.html They are ARML-NYSML Contests 1989–1994 and Leningrad Mathematical Olympiads 1987–1991. The latter is representative of the oldest and most prestigious competition held in the former Soviet Union and the former is the third book of published problems from two highly-regarded competitions in the US. [The first two books are available from Mu Alpha Theta and NCTM.]

The Log regularly published the Power Question from the ARML Contest through 1994. In addition, this contest features individual problems, team problems, and relays. A few of my favorite problems from this book are given below - others will be given in future issues. Some answers and hints are given, but you will profit from working out the details and trying to construct another similar problem. Send me your comments on any of these problems.

• The bisectors of the angles of quadrilateral ABCD determine quadrilateral PQRS as shown.
If $\angle S + \angle P = 193°$, find $\angle B - \angle D$.



Comment: Ans. 26°. A nice problem combining the result that PQRS is a cyclic quadrilateral [its opposite angles are supplementary] with some equation solving.

• Find the number of real values of x that satisfy $\left| \left| x^2 - 1 \right| - 1 \right| = 2^x$. [The vertical bars are absolute value.]

Comment: Calculators were not permitted on this test, but are now permitted on many tests. Sketch the graphs of both sides of the equation using the graphing principles and then check your answer on your graphics calculator.

• An open rectangular box can be formed by cutting identical squares off the four corners of a rectangular piece of cardboard and folding up the four sections that stick out. For a particular sized piece of cardboard, the same volume results when squares of side 1 or squares of side 2 have been cut out. Find the resulting volume when a square of side 3 is cut out.

Comment: Ans. 24 The version of the problem where the dimensions of the rectangle are given and the side of the square yielding the largest such box is to be found is a problem in every graphics calculator manual and many problem solving books. This variation is an interesting application of manipulating an equation in two variables. Can you determine the dimensions of the original rectangle?

• Compute the number of integers between 100,000 and one million with the property that their digits are different and in increasing order [from left to right].

Comment: Ans. 84 A classic problem where finding a certain combination yields the answer even though the order of the digits is insignificant.

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[√At the Root Of It All - continued from page 3]
Sandor is currently working for President Clinton's Panel on Educational Technology investigating innovative ways to integrate technology in the classroom.

Despite their frequent moves, the three have kept in touch and have been able to get together from time to time. Last summer they converged at Dr. George Berzsenyi's Young Scholars Program in Indiana where Sam taught for a month. "The three of us continue to collaborate", Richard comments, "and dream of various ventures, some mathematical and some not, and hope to find more innovative ways to contribute to education."

As Aaron, Sam, Richard, and Sandor have proven, the ties to Mu Alpha Theta are firm and powerful. Maybe their journeys will take them to Orlando this summer!

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Some of the winners in the Texas State Competitions
TEAM TEST: LBJ HS, Austin, Alex Saltman, Edward Early, Darrick Chang, Emily Marcus
SWEEPSTAKES: A&M Cons. HS, College Station
ALGEBRA: Neal Shah, Clear Lake HS
UIL MATH: Damian Burch, Elkins HS
GEOMETRY: Steve Hoberman, LBJ HS
ADVANCED TOPICS: Darrick Chang, LBJ HS
ADVANCED ALGEBRA: Norman Lu, A&M HS
CALCULUS: Alex Saltman, LBJ HS
GEOMETRY [closed]: Brian Chang, Klein HS
SENIOR SCHOLARSHIP TEST: Alex Saltman, LBJ HS
CALCULATOR [Novice]: Norman Lu, A&M HS
MATH TRIVIA/HISTORY: Edward Early, LBJ HS
NUMBER SENSE [Veteran]: Janet Chen, A&M HS
NUMBER SENSE [Novice]: Norman Lu, A&M HS
CALCULATOR [Veteran]: Damian Burch, Elkins HS
ADVANCED TOPICS [closed]: Janet Chen, A&M HS
ADVANCED ALGEBRA [open]: Yivong Shen, LBJ HS

15 The Bulletin Board

• STATE AND REGIONAL MEETINGS in 1996 •
Send information about your state or regional meetings in 1996-97 to the national office so we can list it here.

• FUTURE NATIONAL CONVENTIONS
1997 Seattle
1998 Chicago area

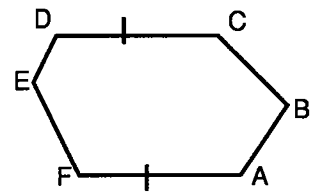
From Chris Shen Student Delegate Sergeant-at-Arms

I am currently a senior at Farragut High School in Knoxville, TN. As far back as I can remember, mathematics has always been my favorite and best subject. From attending the National MATHCOUNTS competition in 8th grade to attending two National MAΘ conventions, I have met many students who enjoy competing AND excelling in mathematics as much as I do. In the past two conventions that I have attended, I noticed the same schools were attending. I think that the reason why we don't have many new schools attending each year is because they do not know enough about Mu AlphaTheta to be willing to attend.

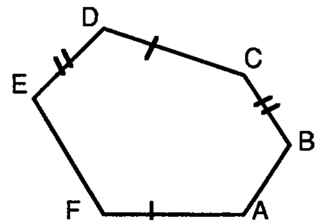
In October, I was asked to be a student host at the Southern Regional Conference of the National Council of Teachers of Mathematics (NCTM). As I was on my break, two math teachers from Mississippi came up to ask me for directions. Soon thereafter, they started asking me information about MuAlpha Theta. Before I knew it, they had asked for my school address and were thinking about the possibility of attending the national convention. Just like that, I had already encouraged a new school to attend!

Although I look forward to seeing the friends that I have made from previous conventions, I also hope to see some new faces too. See you all in Orlando!

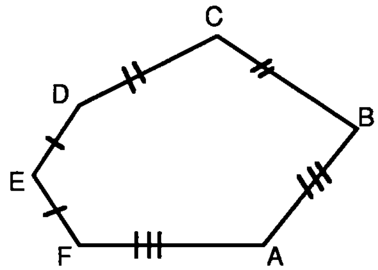
['Tiling the Plane' - continued from page 6]
Convex hexagons that tile the plane:



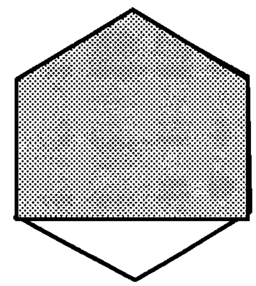
$\angle A + \angle B + \angle C = 360°$



$\angle A + \angle B + \angle D = 360°$



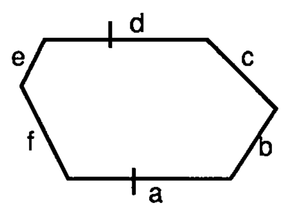
$\angle A + \angle C + \angle E = 360°$.



pentagon

In 1975, Majorie Rice read that all tilings with convex pentagons had been found. Without any special training in mathematics, she found four more types of such pentagons. Still another was found in 1985. The problem of finding all such pentagons, to my knowledge, remains unsolved today.

Generalizing the technique used to show any quadrilateral will tile the plane, Conway's Criterion for determining whether polygon can tile a plane requires that the perimeter of a polygon be divided into six parts a, b, c, d, e, f, and satisfy the conditions:



• two opposite edges, a and d are equal and parallel in the sense that the lines containing these edges are parallel

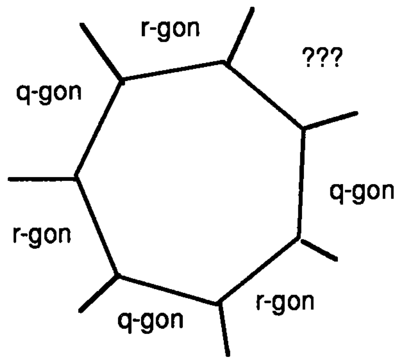
• each of the other four edges b, c, e, and f has a center of symmetry.

Problem 5
a. How can we use Conway's Criterion to see whether a pentagon will tile the plane?
b. Find an equilateral pentagon that will tile the plane.

Problem 6
Use Conway's Criterion to help find a nonconvex hexagon that will tile the plane.

We have just scratched the surface of this fascinating topic. You might want to investigate further on the Internet. See DiaLogue for some suggested sites to try.

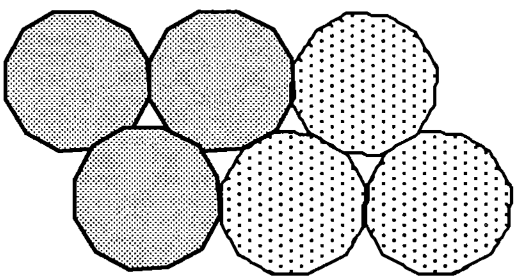
['Tiling the Plane' - continued from page 1]



This uncertainity can be avoided only if the q-gon and the r-gon are the same. Thus in the triple (p, q, r) if p is odd, then q = r. Similarly if q is odd, then p = r and if r is odd, than p = q. This observation reduces our table to

p	q	r
3	12	12
4	6	12
4	8	8
6	6	6

Two of these are Figures 1 and 2 above; (3,12,12) is shown below.



Try to draw (4,6,12), perhaps using Geometric Drawing software.

Problem 1
Repeat a similar analysis to find all tilings where 4 and 5 polygons meet at each vertex.

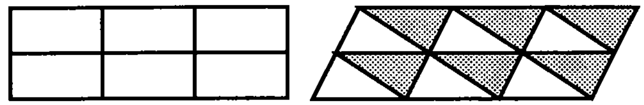
When six polygons meet at a vertex, the equation

$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}+\frac{1}{t}+\frac{1}{u}=2$ has only one solution

p = q = r = s = t = u = 3 – the regular tiling with equilateral triangles. It is impossible for seven or more regular polygons to meet at a point . {Why?]

Tilings with Nonregular Polygons

Analogous to the determination of all regular tilings is the question of what nonregular polygons will tile the plane? Looking at a brick wall shows that rectangles work, but what about other convex polygons? If we 'tilt' a rectangle, we see that any parallelogram will work. Consequently any triangle will work, but we must invert half the triangles.



Consider an arbitrary convex quadrilateral. As with the triangle, take another copy of the quadrilateral, invert it, and join the corresponding edges to form a hexagon [Fig. 3]. The opposite sides of this hexagon are equal and parallel [Why?], so translating this hexagon tiles the plane [Fig. 4] This argument confirms the somewhat surprising result that any [convex] quadrilateral will tile the plane.

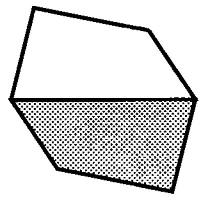


Figure 3

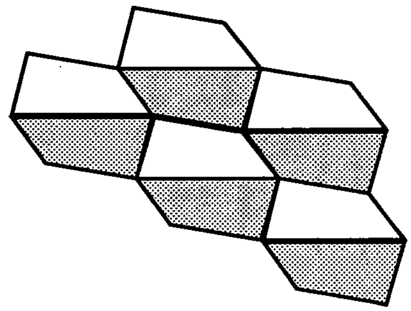


Figure 4

Problem 2 Use this technique to show any nonconvex quadrilateral will also tile the plane.

Problem 3 Using the facts that any at least three polygons must meet at each vertex of a tiling and the sum of the interior angles of a convex n-gon is (n–2)180 to show that no convex heptagon can tile the plane.

The rectangle, the parallelogram and the hexagon above are examples of cells. A **cell** is a region that can be translated to form a tiling.

The result of Problem 3 shows that it remains to check pentagons and hexagons to complete an answer to the question of which convex n-gons will tile the plane.

In 1918, K. Reinhardt found that there were three convex hexagons that tile the plane. They are shown on page 5.

Problem 4 Choose one of these three types of hexagons. Create an example using side lengths and angle measures that fulfill the indicated conditions. Cut out several copies of your hexagon and make a tiling. How many of your hexagons are needed to make a cell?

If we cut off a triangle from a regular hexagon, we form a convex pentagon that will tile the plane. [Why?] There are fourteen known types of convex pentagons that will tile the plane.

[continued on page 5]