

The

# Mathematical Log

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## Areas of "Cevianal" Polygons - Making Something Out of Nothing

This article was prompted by a letter to the editor in the Mathematics Teacher [Dec. 1994, p. 726] by Ryan Morgan, a ninth-grade geometry student from Baltimore, Maryland. His discovery was also chronicled in several newspapers. We shall, however, begin at a mathematical beginning and take a geometrical excursion that includes Ryan's discovery. Along the way we will offer some comments on making generalizations, examples of using different methods of proof, and possible challenges for you to pursue.

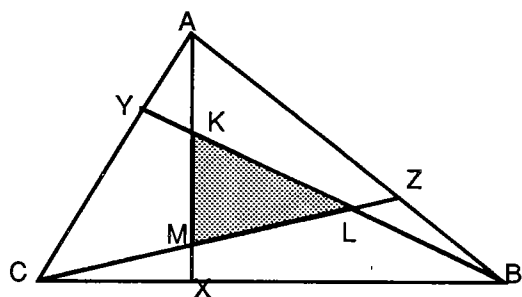
To begin, recall that the medians of a triangle are concurrent in its centroid. To generalize this familiar theorem, we could try several approaches:

- medians bisect the sides of the triangles; try trisectors, "quadsectors", etc.
- try the analog of a median for other shapes such as quadrilaterals
- try cevians other than medians, say angle bisectors. A cevian is a segment from a vertex of a triangle to a point on the opposite side.

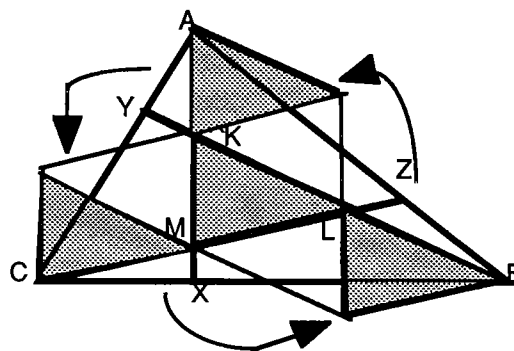
We will, in fact, explore all three approaches. As we do, you are encouraged to use your geometric drawing software such as The Geometer's Sketchpad, The Geometric Supposer, etc.

Let  $S(ABC)$  denote the area of  $\triangle ABC$ ;  $S(ABCD)$  denote the area of quadrilateral  $ABCD$ , etc.

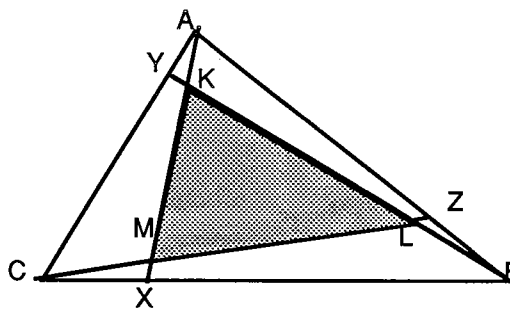
**Problem 1:** Segments  $AX$ ,  $BY$ , and  $CZ$  are trisectors of sides  $BC$ ,  $AC$ , and  $AB$  respectively of  $\triangle ABC$ . What is the ratio  $\frac{S(KLM)}{S(ABC)}$ ?



**Solution:** The dissection proof below shows the answer to be  $\frac{1}{7}$ . Explain how the dissection works. [See also the cover of the Mathematics Teacher, Feb. 1992]



**Problem 2:** Segments  $AX$ ,  $BY$ , and  $CZ$  are quadrisectors of sides  $BC$ ,  $AC$ , and  $AB$  respectively of  $\triangle ABC$ . What is the ratio  $\frac{S(KLM)}{S(ABC)}$ ?



**Solution:** Consider the case of an equilateral triangle. [See figure on page 6] If each of the three small corner triangles has area 1, then the three triangles labeled '3' each has area 3 since they have the same height as those labeled '1' and their bases are three times as long. The areas of the other triangles are denoted by  $x$  and  $y$  as shown. We seek the ratio  $\frac{y}{y + 3x + 12}$ . Now  $x + y + 3 = 3(x + 1)$  and  $2x + y + 7 = 3(x + 5)$  [Why?]. Solving these equations yields  $x = 8$  and  $y = 16$  showing that the desired ratio is  $\frac{4}{13}$ . [continued on page 6]

# dia Log ue

with Log Editor Tom Butts

Richard Rusczyk sends along this puzzle about famous mathematicians.

E D E S F N E P Y T H A G O R E A  
 N U Z Q E A V A N D E R V E L D E  
 T R C W R P T S S S P I W D I R B  
 A C T L M I Y J O Y C D I L C U E  
 M O R O A E O E T T E A L A C E R  
 N S E T R R O H K S L U E M A N N  
 R I A G D B A O C C O S S L N O O  
 T N T A A G L C A N O B I F O T L  
 E E N R O S A E R L H H E E B N L  
 S E T R A C S E D E L C S R I I G  
 K L A N R T B E U B U U R M F L A  
 O S B L E D N A M L A C S A P C U  
 G U M P L S S U A G E M L T C R A  
 A R C H I M E D E S L R T I E R R

The names of 14 famous mathematicians are hidden in the grid above. Match the names with the descriptions below.

1. While in grade school, he discovered and proved a formula for the sum of the first n positive integers.
2. The e of natural logarithms was named in his honor; some of his 750 articles and books were prepared while he was blind.
3. "I think, therefore I am" philosopher who also made great contributions to the development of coordinate geometry.
4. His proof for his Last Theorem, alas, could not fit in the margin of his book; so we had to wait over 300 years to finally see the theorem proved.
5. One of five brothers who all made significant contributions to mathematics and science.

6. This pioneer of fractal theory loaned his name to one of the most famous fractal and a high school mathematics competition.
7. To some, an outdated computer language; to others, a triangular array of numbers discovered in China.
8. Succeeded where #4, and thousands of others, failed.
9. His 'bones' were a rudimentary computer of logarithms.
10. His secret numerological society based on integers and their ratios was crushed by the discovery that the square root of 2 was not rational.
11. He reportedly burned the sails of attacking Roman ships with lenses he developed. The Romans got their revenge when a soldier killed him after being scolded for messing up the mathematician's circles.
12. This apple lover developed many fundamentals of calculus en route to describing principles of physics.
13. 1, 1, 2, 3, 5, 8, 13, 21, ... : a sequence of numbers containing the fundamental principles of nature.
14. His "Elements" have been used by teachers to inspire future mathematicians for millennia, including being used by Mel Gibson in "Man Without a Face".

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You are encouraged to send your comments, suggestions on what type of articles and features to include in the Log, a question about any area of mathematics - a person, a concept, a problem,..., news of your chapter, or anything else on your mind. Send it to me at the address on page 4.

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This is your last chance to think about attending the national convention. Highlights include a spectacular day of white water rafting on the famed Kennebec River in the wilderness of western Maine. A spectacular picnic will be held two-thirds of the way up the mountain - we can climb to the summit and ride down using the lift. Two bonfire dances are also planned.

All testing will be completed by mid-afternoon with intramurals, cribbage, bridge and other tournaments following. Write to Pete Pedersen, P.O. Box 801, Granite Island RD, Vinalhaven, ME 04683 or to the national office for more information.

Pete is profiled in  $\sqrt{\text{At the Root Of It All}}$

# 1995 $\sqrt{\text{At the Root of It All}}$

*Deborah Patonai Phillips, Activities Editor*

St. Vincent-St. Mary HS, 15 North Maple Street, Akron, OH 44303

As Mu Alpha Theta sponsors, we all face numerous hurdles and challenges unique to our particular situations. One inspiring sponsor lives and works on an isolated island but remains a major force in state mathematics. He is Lee Travis Pedersen, known affectionately to everyone as Pete. Nominated by his students and chosen by his fellow sponsors, Pete was selected as the Huneke Award winner at last year's national convention. Named after former Secretary-Treasurer Harold Huneke, this award is given annually to an outstanding Mu Alpha Theta sponsor.

$\sqrt{\text{At the Root Of It All}}$  is proud to recognize Pete's many accomplishments.

Pete has been teaching for 18 years on Vinalhaven Island, a small island seventeen miles off the rocky coast of Maine. With a population of approximately 1100, the high school has only 45-50 students in grades 9-12. Over the years, Pete has taught everything from junior high English, mathematics, and social science to high school history, chemistry, physics, computer science, Maine studies, psychology, and, of course, mathematics. Whatever the subject, Pete's knowledge and enthusiasm makes it interesting.

In the mid 1980's, Pete introduced an integrated mathematics program. He blended arithmetic, algebra, and geometry into a two-year program. The third year contains work on logarithms, trigonometry, and discrete mathematics. During the fourth year, students may take AP Calculus or a fourth year of blended mathematics. The results have been astounding. Students who have participated in the program for four years score, on the average, 100 points higher on the SAT test than their predecessors. As two of his former students put it, "As a result of Pete's innovation, the mathematics curriculum has improved and the students are more interested and enthusiastic about learning it."

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**"Contributions came from around the world because just as Pete believes in his students, others believe in him and his efforts."**

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In 1987, Pete instituted Math Team as an extracurricular activity. That year, Pete's team placed last out of 91 Maine schools. Vowing that 'this would never happen again', he began a class aimed at improving the abilities of his students as test-takers, competitors, and cooperative problem solvers. With his unique approaches, Pete transformed his team into one of the best in the state. Now more than a quarter of the students participate and several have placed on

the all-state team. "For those on the Math Team, improvement in competition has meant a similar improvement in the classroom", according to former students.

About this same time, Pete was introduced to Mu Alpha Theta. He began to take his team to both regional and national conventions. Offering his students a chance to visit many parts of the country as well as compete mathematically, these trips are very important to Pete - so important that he finances them! For the sizable expenses for the Hawaii convention, Pete raised money by organizing beanos, casserole suppers, and pledge drives. "Contributions came from around the world because just as Pete believes in his students, others believe in him and his efforts."

Pete's dedication to his students has inspired them to learn and excel. His students know they can meet with him "anytime". It is not unusual to him to be working with students as late as midnight or at 5:00 a.m. His door is always open to his students, even during the summer. Two of his former students, M. Webster and R. Jenner, describe their friend and teacher: "Pete's availability and responsiveness show students that he is devoted to mathematics and to them. He tailors the speed and intensity of the student's progress through the program - each student is in a class of one. Students have more responsibility for their learning - they feel they are true partners in the process."

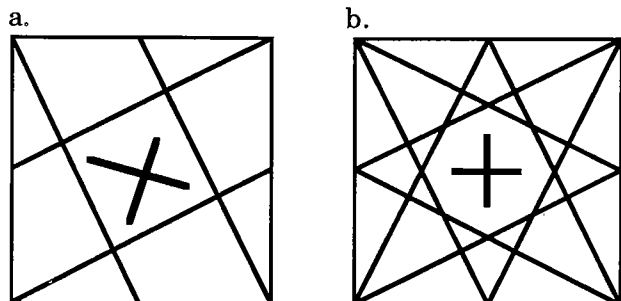
Pete's teaching schedule might discourage even the most dedicated teacher. Teaching six of the seven periods, he often has two different classes per period, from junior high to senior high level. "He not only flows from one class to another, but from one student to another." Each class, from the most basic to the most advance, receives the same amount of Pete's attention. Pete has a gift to be able to motivate students at all levels.

Besides being a teacher and math team coach, Pete has become a major force in Maine mathematics organizations. The two Mathematics Leagues in which he was an officer have become the largest and most influential of the seven leagues in Maine. Pete is also chief coordinator of the Maine State Meet which is attended by over 90 schools. He has also assumed the coaching position for the Maine All-Star Team made up of the twenty highest-scoring competitors from the regular season. This team is Maine's delegation to the ARML. [continued on page 5]

[Areas of Cevian Polygons , continued from p.6]

Generalizing to quadrilaterals leads to a familiar puzzle [part a of Problem 4]:

**Problem 4** Compute the ratio of the area of the region with a cross in the middle to the area of the square. The segments from a vertex of the square to the midpoint of the opposite side play the role of a median of a triangle. The trivial case of concurrence [analogous to the concurrence of the medians of a triangle] holds when the segments are diagonals.

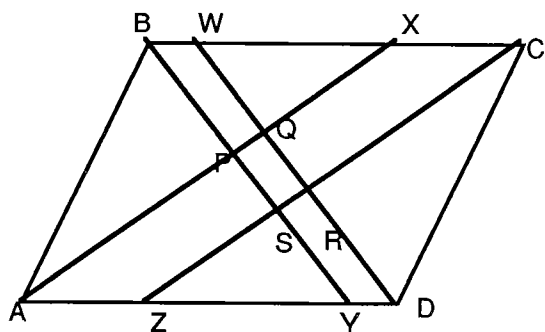


**Exercise 4:** Show the answer to part a is  $\frac{1}{5}$  using a dissection proof and the answer to part b is  $\frac{1}{6}$  using a coordinate proof.

**Exercise 5:** Show the results in Exercise 4 are valid for if you begin with a parallelogram and draw segments from a vertex to the midpoint of the opposite side.

**Exercise 6** Try to formulate and solve some analogous area problems involving other shapes.

Finally what happens if we consider angle bisectors and trisectors instead of medians, trisectors of sides, etc.?



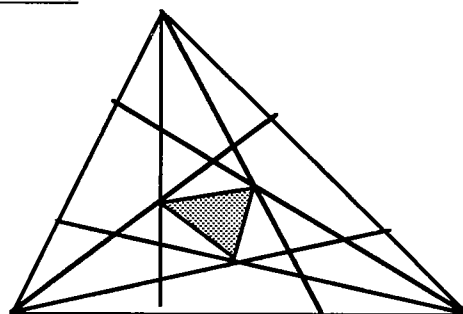
**Exercise 7** ABCD is a parallelogram with  $BC > CD$ . AX, BY, CZ, and DW are angle bisectors.

a. What type of quadrilateral is PQRS? Justify your answer.

b. If  $BC = a$ ,  $CD = b$  and  $\angle BAD = \phi$ , what is the ratio of the area of PQRS to the area of ABCD in terms of a and b? [Show it is independent of  $\phi$ ]

As with the medians of a triangle, the angle bisectors of a triangle are concurrent. What happens if we consider the figure formed by the trisectors of the angles of a triangle?

**Problem 5**



What is special about the shaded triangle ?

**Exercise 8** Answer the question in Problem 5 by doing some research on Morley's Theorem.

**Exercise 9** Investigate area relationships involving the shaded triangle, the hexagon formed by the angle bisectors, and the original triangle.

You might send a letter to the editor of the Mathematics Teacher to describe any discoveries you make.

## The Mathematical Log

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The Mathematical Log is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to Mu Alpha Theta National Office, 601 Elm Ave., Room 423, Norman, OK 73019. or to Log editor Thomas Butts, Univ. of Texas at Dallas, P.O. Box 830688 FN 32, Richardson, TX 75083  
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## MA $\Theta$ Bulletin Board

From Kevin Stancoven  
Student Delegate Sergeant-at-Arms

• STATE AND REGIONAL MEETINGS in 1994-95 •

April 1995: Mississippi State Meeting. Jackson, MS

April 28,29 1995: Florida State Meeting.

• FUTURE NATIONAL CONVENTIONS

1996 - Orlando Area - sponsored by Miami Sunset SR HS, Miami Killian SR HS, Miami Springs SR HS

1997 ???

1998 Chicago area ??

Send information about your state or regional meetings to the national office.

**Sponsors:**

• The Mu Alpha Theta Breakfast at the NCTM Annual Meeting in Boston will be held Friday, April 7, 7:00 a.m. at the Boston Back Bay Hilton. Cost is \$7.00 per person which can be sent to the national office.

• Remember dues are now **\$3** per student.

• The new student delegate officers want to encourage chapters to correspond with each other, especially experienced chapters with new chapters. Contact the national office if your chapter wishes to participate.

**Students:**

• Did you know that you can purchase MA $\Theta$  jewelry, T-shirts, and books directly from the national office? Pins are \$7.00; charms, \$6.50; buttons or tie tacks, \$6.00; patches, \$1.50; medallions, \$4.00, and T-shirts, \$8.00. Write to the national office for the list of available books or to place an order.

• An idea for graduation --- become a "Friend of Mu Alpha Theta". Receive the Math Log regularly and any new publications for just \$10 a year.

### In Memorium

We were saddened to learn of the death of Josephine Andree in February. While attending the University of Chicago, she met Richard Andree and they married on her graduation day in 1944. They founded Mu Alpha Theta at the University of Oklahoma in 1957. She edited nearly 60 books that her husband wrote. The new handbook for Mu Alpha Theta sponsors will be dedicated to her.



My name is Kevin Stancoven and I am currently a junior at West Orage-Stark HS in Orange, TX. Since mathematics has always come easy for me, I started participating on the Math Team in the sixth grade. We often have a hard time, at our school, finding more than five people to attend the tournaments in our area - I am working on ways to increase this number. To increase attendance at the national convention, we could try to hold some conventions in the midwest so that more schools from the western half of the country could participate without needing to travel thousands of miles. We could also try to find ways to get more publicity for Mu Alpha Theta, especially in areas where there are currently very few chapters. See you in Maine!

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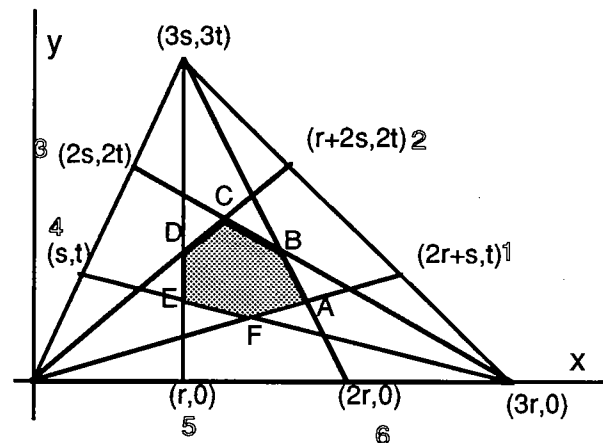
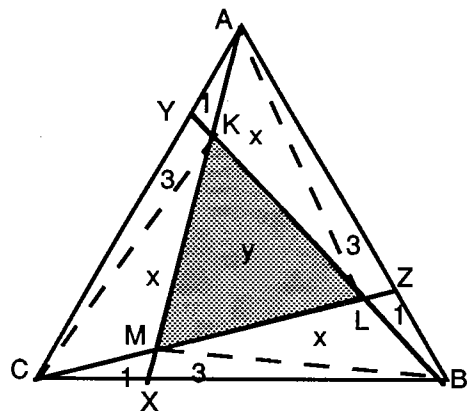
[√ At The Root Of It All continued from page 3]

Pete is also working hard at promoting mathematics at the pre-secondary level by forming an Inner-Island Math League for grades 4-8. In his "spare" time, Pete is working to preserve his special island community. He is active in several civic organizations and the church. Because of his devotion to his students, Pete has been recognized by his peers. In addition to the Huneke Award, he has been a finalist for Maine Teacher of the Year and in 1993 was an awardee of the Presidential Award for Excellence in Mathematics Teaching for Maine.

This summer Pete is serving as the chairperson for the national convention. He wants all students to have success in the competitions and have a good time as well.

Pete has truly influenced the lives of many students in the past 17 years. Mu Alpha Theta is proud to have him as a sponsor and a friend.

[Areas of Cevial Polygons , continued from p.1]



**Exercise 1** Show that if each of the ratios  $\frac{AY}{YC} = \frac{CX}{XB} = \frac{BZ}{ZA} = \frac{1}{n}$  then the analogous area ratio is  $\frac{(n-2)^2}{n^2-n+1}$ .

The results are given below; the algebraic calisthenics are left to the reader.

**Exercise 2** Do some research on Routh's Theorem which states that if  $\frac{AY}{YC} = \frac{1}{k}$ ,  $\frac{CX}{XB} = \frac{1}{m}$ ,  $\frac{BZ}{ZA} = \frac{1}{n}$  then the analogous area ratio is

The equations of the six lines are:

1:  $y = \frac{t}{2r+s}x$     2:  $y = \frac{2t}{r+2s}x$     3:  $y = \frac{2t}{2s-3r}(x-3r)$

4:  $y = \frac{t}{s-3r}(x-3r)$     5:  $y = \frac{3t}{3s-r}(x-r)$

6:  $y = \frac{3t}{3s-2r}(x-2r)$ .

$$\frac{(kmn - 1)^2}{(km + k + 1)(mn + m + 1)(nk + n + 1)}$$

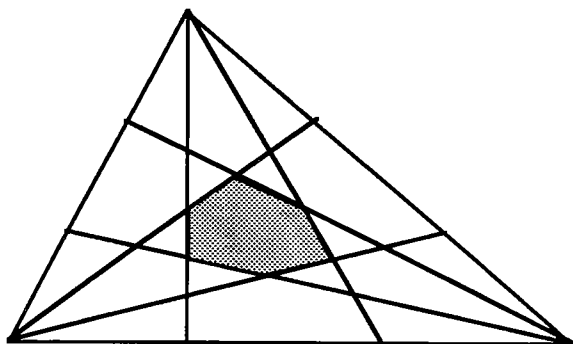
These lead to the coordinates for each of the six vertices:

What are the special cases of Routh's Theorem considered in Problems 1 and 2?

A:  $(\frac{3(2r+s)}{4}, \frac{3t}{4})$     B:  $(\frac{6(r+s)}{5}, \frac{6t}{5})$     C:  $(\frac{3(r+2s)}{4}, \frac{3t}{2})$

D:  $(\frac{3(r+2s)}{5}, \frac{6t}{5})$     E:  $(\frac{3(r+s)}{4}, \frac{3t}{4})$     F:  $(\frac{3(2r+s)}{5}, \frac{3t}{5})$

**Problem 3** The first case of Ryan's discovery comes from considering the ratio of the area of the hexagon formed by taking both trisectors from each vertex of a triangle to the area of the triangle. [see Mathematics Teacher, Nov. 1993, p. 619; Dec. 1994, pp. 726,743]



Set  $A = (x_1, y_1), \dots, F = (x_6, y_6)$  and calculate the area

$$S(ABCDEF) = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_1 \end{vmatrix} \text{ to}$$

show that  $S = \frac{9rt}{20}$  and the desired ratio is  $\frac{1}{10}$ .

**Exercise 3** What are some properties of hexagon ABCDEF that you can obtain using the coordinates of the vertices?

Ryan generalized this result by conjecturing that if we consider the hexagon determined by the two "middle-nectors" from each vertex of the triangle where n is odd, then the ratio of the area of the hexagon to the area of the triangle is  $\frac{1}{z}$  where  $z = \frac{9n^2-1}{8}$ . Can you prove it?

[continued on page 4]

**Solution:** We use a coordinate approach and the Coordinate Area Formula [see Log, Dec. 1990]. The strategy is to find the coordinates of the vertices A to F by viewing them as points of intersection of certain pairs of lines labeled 1 to 6. Then apply the Coordinate Area Formula to find the area of ABCDEF and compare it to  $\frac{9rt}{2}$ , the area of the triangle.