Attend the 25th Annual Convention in Maine 7/28 - 8/3

The

Mathematical Log

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Bernoulli's Binomial Rule

In the last issue, we asked these three questions in a test about preconceptions concerning probability.

> 1. The probability of a baby's being a boy is about $\frac{1}{2}$. Over the course of an entire year, where would there be more days when at least 60% of the babies born were boys?
> A) in a large hospital B) in a small hospital C) makes no difference
>
> 2. A large bag has the same number of black balls and white balls in it. Which is more likely?
> A) Choosing 7 whites in 10 draws B) Choosing 70 whites in 100 draws C) same likelihood
>
> 3. The probability of a baby's being a boy is about $\frac{1}{2}$. What is the probability that among six children, [exactly] three will be girls. A) less than $\frac{1}{4}$ B) between $\frac{1}{4}$ and $\frac{1}{2}$ C) $\frac{1}{2}$ D) more than $\frac{1}{2}$

The answers were 1. A 2. A 3. B. Many people chose C for both #1 and #2 erroneously feeling that the size of the sample has no effect on the probability. To help understand the idea behind all three problems, we turn to a famous result of Jakob Bernoulli in his masterpiece of probability theory: In Ars Conjectandi.

"From the 1670's until the end of his life, [Jakob] Bernoulli was one of the world's leading mathematicians. His remarkable talent was coupled with a prickly personality, a massive ego, and a tendency to belittle the efforts of those less gifted. ..... His productive years coincided with Gottfried Wilhelm Leibniz's discovery of the calculus. Jakob was one of the chief popularizers of this immensely fruitful subject. ... In this undertaking, he had an uneasy ally in Johann, the younger of what we might call, with unashamed alliteration, the brilliant but bickering Bernoulli brothers. Jakob, in fact, had played a role in teaching mathematics to his younger sibling. In later years, he probably regretted that he taught Johann so well, for the youngster turned out to be a mathematician to rival, if not surpass, his teacher. There arose between the brothers a fierce competition for mathematical supremacy. Johann expressed unconcealed glee when he solved a problem that had stumped his brother, whereas Jakob demeaningly called Johann his "pupil" implying that Johann was only parroting his mentor's brilliance." [see [1]].

Exercise 1: Look up the contributions of the Bernoulli brothers to the study of (1) the catenary, (2) the isopermetric problem, and (3) l"Hopital's Rule .

Back to our subject. A Bernoulli trial is an experiment with only two outcomes. It results in heads or tails, boy or girl, on or off, etc. In a Bernoulli trial, one outcome is considered a "success" and the other a "failure". One such trial is rarely interesting. When several trials are conducted, however, an interesting pattern emerges. Those trials must be independent, the outcome of one trial must have no impact on the outcome of other trials. Tossing a coin several times is a sequence of Bernoulli trials, each having probability $p = \frac{1}{2}$ of success. If success is rolling a '3' when rolling a standard die, then the probability of success, p, is $\frac{1}{6}$ and the probability of failure, $q = 1 - p$, is $\frac{5}{6}$. What is, however, the probability of tossing [exactly] 3 heads in 5 tosses of a coin? 7 heads in 10 tosses? 70 heads in 100 tosses? Several typical sequences of five tosses of a coin with 3 heads are HHHTT, HTHTH, and HHTTH. Each of them has probability of $(\frac{1}{2})^5 = \frac{1}{32}$. The number of such sequences is the binomial coefficient $\binom{5}{3}$ since we must "choose" which three of the five tosses will be heads. Thus

$$\Pr(\text{3 heads in 5 tosses of a coin}) = \binom{5}{3}(\tfrac{1}{2})^5 = \frac{10}{32} = .3125.$$

$$\Pr(\text{7 heads in 10 tosses of a coin}) = \binom{10}{7}(\tfrac{1}{2})^{10} = .1172$$

$$\Pr(\text{70 heads in 100 tosses of a coin}) = \binom{100}{70}(\tfrac{1}{2})^{100} =$$

.0002317. [On a TI-82, 100 [n C r] 70 [x] .5 [^] 100; [n C r] is on the MATH menu under PRB]. Even though the probability of success is the same in the second and third cases, the number of tosses makes a big difference.

[continued on page 6]

dia Log ue

with Log Editor Tom Butts

Join us at the 25th Annual Convention at the Sugarloaf Resort in the Carrabassett Valley of Western Maine July 28 – August 3. Vinalhaven High and sponsor Pete Pedersen are the hosts. The cost of $460 per person includes transportation to and from Portland with a stop at LL Beans, lodging for six nights, meals, and white water rafting. Write to Pete Pedersen, P.O. Box 801, Granite Island RD, Vinalhaven, ME 04683 or to the national office for more information.

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You are encouraged to send your comments, suggestions on what type of articles and features to include in the Log, a question about any area of mathematics - a person, a concept, a problem,... , news of your chapter, or anything else on your mind. Send it to me at the address below.

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The October issue contained a picture of Debbie Phillips and her MAΘ team at the New Orleans convention. Unfortunately the computer 'gobbled up' the accompanying text that said that the team won 7 of the top 10 awards in the first-ever History of Mathematics Exam. Congratulations!

****

Several students from Bryan Station HS in Lexington, KY passed along their experiences with a mathematical experiment that your chapter may wish to try.

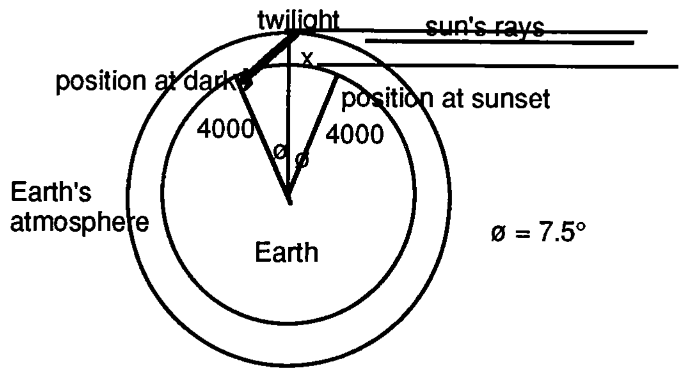
Made possible by the Earth's atmosphere, twilight is the time between sunset and the actual darkness of night. The gases that surround the Earth allow us to see the sun's rays after we can no longer see the sun. Several centuries ago, a scientist named Alhazen concluded that by measuring the length of twilight, he could estimate the height of the Earth's atmosphere.

Last Spring, Mrs. Charlene Norman's calculus class, supervised by our student teacher Mark Payton, attempted to replicate Alhazen's experiment . [Mrs. Norman is the MAΘ sponsor.] We measured the length of twilight, estimated the orientation of the sun's rays, the rotation of the Earth, and other factors to arrive at our conclusion.

Several students measured the length of twilight for several nights and came up with an average measurement of one hour. . We used this time to determine the rotation of the Earth using the proportion

$$\frac{1 \text{ (hours of twilight)}}{24 \text{(hours in a day)}} = \frac{x}{360^\circ}.$$

The Earth rotated 15° during the one hour of twilight.



The radius of the Earth is approximately 4000 miles. We bisected the 15° angle to form a right triangle whose hypotenuse was the distance from the center of the Earth to the top of its atmosphere and x is the height of the atmosphere. Then using the definitioin of cosine of an acute angle

$\cos 7.5^\circ = \frac{4000}{4000 + x}$ so that x = 34.5158 miles.

The students who participated were Kimberly Branham, Carrie Paddock, Nicole Lee, Segun Embry, and Tony Woodson.

The Mathematical Log

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The Mathematical Log is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association ofAmerica (MAA) and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to Mu AlphaTheta National Office, 601 Elm Ave., Room 423, Norman, OK 73019. or to Log editor Thomas Butts, Univ. of Texas at Dallas, P.O. Box 830688 FN 32, Richardson,TX 75083

1994 √At the Root of It All

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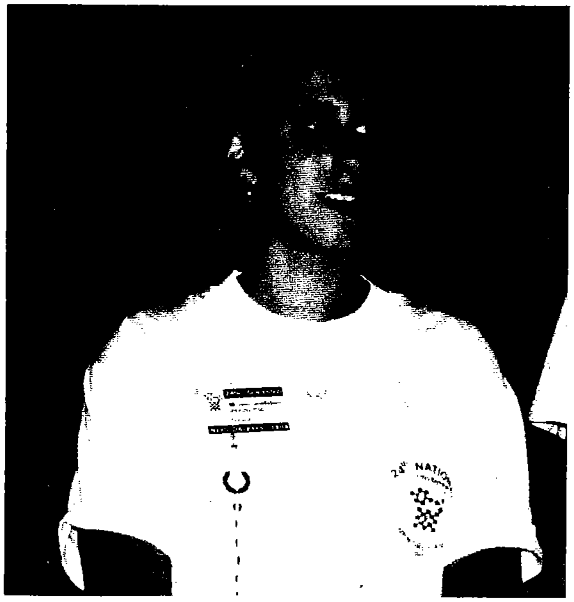


Since the time of Hypatia, the first woman mathematician, women have changed the course of mathematics. They are contributing to the field of mathematics in traditional and in unique ways. Some begin their influence when they are very young. √At the Root Of It All has discovered two such young ladies who are surely on their way to mathematical stardom.

Every summer Mu Alpha Theta honors graduated members with two distinguished awards. The Andree Award, named after Richard and Josephine Andree, the founders of MAΘ, recognize a student who wants to become a mathematics teacher. The Kalin Award, named after former governor and president of MAΘ, Robert Kalin, commends a student for his/her mathematical ability and unusual service to MAΘ. Congratulations to Kalin winner Miriam Goldstein and Andree winner Derinda Tyler.

From Region III, Miriam is a graduate of Berkeley Preparatory School in Tampa, FL. Always involved in Mu Alpha Theta, she served as chapter treasurer as a junior. Responsible for running the school's lunch program for grades 6 – 12, the chapter's main fund-raiser, she used most of the profits to send members to the national convention. As a senior, she was elected chapter president where her main duties were registering her teams for competitions and assisting her sponsor in planning overnight trips to state and national competitions.

In addition, she was solely responsible for organizing and supervising all team math practices scheduled twice a week for two hours after school. Miriam worked separately with each team, helping them with problems and techniques. Even during her free period three times a week, she worked with the Algebra II Team. Her efforts were well rewarded: the Algebra Team placed consistently in the top three in various competitions and won second place Theta at the state convention. According to her sponsor, Thom Morris, "Many students have told me that Miriam really made a difference in their understanding of a difficult topic. She does not talk down to them; she works with them step-by-step until they understand the material. ... She is one of the main reasons that our math club is as strong as it is." Along with her chapter offices, Miriam was elected student delegate president at the Hawaii convention. She also wrote a column for each issue of the Log.

In addition to her leadership abilities, Miriam is also quite a competitor, winning several awards at local, state, regional, and national competitions. "She is the sole reason that our Calculus Team placed first at our state tournament this year," says Coach Morris. She has also won numerous awards at national competitions.

Miriam' abilities are not only math-related. She has earned several top awards in Spanish, physics, chemistry, and history. Also an accomplished pianist, she recently took up the viola, performing as a classical violist and a country fiddler. She has also won three prestigious awards: the MIT Alumni Award for mathematics and science, the Harvard Award for scholarship, and one of 75 semifinalists for the U.S. Physics Team.

"MAθ has provided me with a taste of teaching and a chance to discover the enjoyment and satisfaction of mathematics teaching as a profession."

This year's Andree Award recipient, Derinda Tyler, is a graduate of Austin HS in Decatur, AL. Her father, a mathematics professor and wonderful role model, exposed her to mathematics and its beauty. She was "adding and subtracting before saying the alphabet." Since she was very young, she has wanted to become a mathematics teacher.

Mu Alpha Theta gave Derinda a taste of her dream.
[Continued on page 5]

CONTEST CORNER

This month's column contains the an edited version of the solution to Power Question from the 1994 ARML Contest by Gil Kessler and Larry Zimmerman, the writers of the contest. The questions appeared in the October 1994 issue of the Log, but are repeated here. Following the solution is more information about Pick's Theorem for the area of a lattice polygon – the basis for many of the parts of the Power Question.

Recall: The letter L will refer to the length of a lattice segment as well as the segment itself and N(L) will represent the number of lattice points on segment of length L ,NOT INCLUDING ITS ENDPOINTS.

SEGMENTS

1. Compute N(L) for the lattice segment L whose endpoints are a. (0,0) and (3,4) b. (0,0) and)12,16) c. (0,0) and (300,400) d. (–3,11) and (57,101) .

a) 0 b) 3 c) 99 d) 29 In general, if the endpoints of L are (a,b) and (c,d). N(L) = gcd[(a–c),(b–d)]

2. Compute all possible values for N(L) if L is oblique and
 a. L = 13 b. L = 4·13 c. L = 5·13

a. Any positioning for an oblique segment with L = 13 is equivalent to positioning it from (0,0) to (5,12), so N(13) = 0.
b. Extending L to four times its length gives N(52) = 3.
c. This segment is 5 times a segment of length 13; N(L) = 4, **or** 13 times a segment of length 5; N(L) = 12, **or** a hypotenuse of a primitive right triangle [either 13–63–65 or 33–56–65]; N(L) = 0. The sides of primitive right triangles can be found using the well-known expressions $m^2 - n^2$; $2mn$, $m^2 + n^2$.

3. Compute the number of different possible values for N(L) if L is oblique and L = 5·13·17.

There are 7 values. Some cases are: (1) 13·17.– 1 = 220 copies of segment of length 5; 17– 1 = 16 copies of segment of length 65; and 0.

Pick's Theorem [1899] The area of any lattice polygon is given by $K = \frac{1}{2}B + I - 1$ [where K is the area, B is the number of lattice points on the boundary (sides and vertices) of the polygon, and I is the number of lattice points in the interior of the polygon.]

POLYGONS

4. Compute the value of I for the triangle whose vertices are (1,2), (31,42), and (81,–78).
A = 2800 [use determinants or Hero's Formula; B = 100, so I = 2751.

5. a. Compute the maximum value of I for a lattice square whose side is 25.
624: To maximize I, minimize B by placing two vertices at (0,0) and (24,7).

b. Compute the maximum value of I for a lattice square whose side is 26.
673: The only other value of I is 625.

6. Show that if two sides of an oblique lattice triangle are 5 and 13, then the third side cannot be an integer.
The third side must be between 13 – 5 and 13 + 5. Checking shows the areas of all such triangles are irrational, but Pick's Thm. guarantees the area of a lattice triangle is rational of the form $\frac{k}{2}$.

7. a. Given any 5 lattice points, show there is a segment joining two of them that contains a lattice point (other than the endpoints of the segment).
Given any five lattice points, at least two of them must have coordinates of the same parity, such as (odd,even). The midpoint of the segment joining these two points is also a lattice point.

b. Prove that every convex lattice pentagon must contain an interior lattice point.
Using the argument of part a, two of the vertices of the pentagon must have the same parity. If they are non-adjacent, their midpoint is an interior lattice point; if they are adjacent, show there must be another lattice point on that side of different parity and it can be joined to another vertex to produce a midpoint that is an interior lattice point.

c. A "trapezium" is a quadrilateral with no pair of parallel sides. For a convex lattice trapezium:
i. Show that if two successive sides each contain a lattice point (other than the vertices), then there is a lattice point in the interior of the trapezium.
If Q and R are the lattice points on sides AB and BC respectively, then pentagon AQRCD has an interior lattice point by 7b.

ii. Show that if there are two opposite sides neither of which contain a lattice point (other than the vertices), then there is a lattice point in the interior of the trapezium.
Try to produce a lattice pentagon that lies inside the trapezium.

8. If the product of two sides of a lattice triangle is a prime, and the area is also a prime, prove that the area must be 2.
There are two possibilities: sides are $[\sqrt{p}, \sqrt{p}, s]$ or [1,p,t] . Using $A = \frac{1}{2} p \sin ø$, the Law of Cosines, and the formulas for the sides of primitive right triangles given in 2c, show that are four such triangles with p = 5.

The formula for the area of a lattice polygon is a gem of elementary mathematics. It seems to be the only claim to fame of Gyorgy Pick. To discover this theorem, as Pick may have done in 1899, we can apply the problem solving strategy of considering simpler versions of the problem. Examine lattice polygons with no interior points; in particular, rectangles with one side of 1.

Size	B	I	A
1x1	4	0	1
1x2	6	0	2
1x3	8	0	3
1x4	10	0	4
1xn	2(n + 1)	0	n

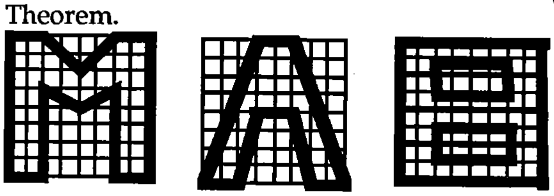
[continued on page 5]

[continued from page 4]
Thus B is a little more than twice A, in fact, $B = 2A + 2$ or $A = \frac{B}{2} - 1$. To see the effect of I, the number of interior points, consider a series of rectangles with 1, 2, 3, ..., n interior points.

Size	B	I	A
2x1	6	0	2
2x2	8	1	4
2x3	10	2	6
2x4	12	3	8
2x5	14	4	10
2xn	2(n+1)	n−1	2n

In each case, the difference $A - [\frac{1}{2}B - 1] = I$, so it is plausible that $A = \frac{1}{2}B + I - 1$.

Exercise: Find the sum of the three areas using Pick's Theorem.



There have been several attempts to generalize Pick's Theorem. The first deals with figures like the 'theta' above – lattice polygons with 'holes' in them.

For any lattice polygon with lattice holes, let B be the number of boundary points including the boundary points of each hole, I be the number of interior points of the polygon [not including the interior points of the holes - a hole has no interior points], and H be the number of holes. See if you can prove the

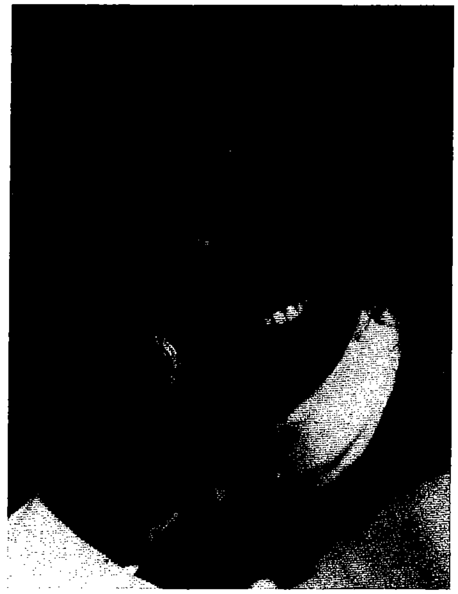
Extended Pick's Theorem: $A = \frac{1}{2}B + I + H - 1$.

Find examples with H = 1, 2, 3, ... Use Pick's Theorem in your proof of this theorem.

Another attempt was to find an analog of Pick's Theorem for [three-dimensional] lattice polyhedra. In 1957, John Reeve proved that no such formula could exist by considering pyramids with vertices at (0,0,0), (1,0,0), (0,1,0), and (1,1,z). Check the cases z = 1 and z = 2 to understand Reeve's counterexample.

References:
1. Another Fine Math You've Got Me Into, Ian Stewart, Freeman, 1992, Chap. 5.
2. "Chopping Up Pick's Theorem", N. Vasilyev, Quantum, Jan./Feb. 1994, pp. 49 – 52.

[$\sqrt{\text{At the Root Of It All}}$, continued from page 3]
Tutoring both junior high and senior high students in mathematics, she soon discovered that she could "explain math in a way that people understood." While serving as vice-president of her MAΘ chapter, Derinda conducted an after-school tutoring program at one of the city's middle schools. As a result of this program, the school nominated the Austin HS MAΘ chapter for a volunteer service award.



Besides MAΘ , Derinda was also involved in other scholastic activities as well as athletics and music. As a leader, she was president of the National Honor Society, vice-president of the Chemistry Club, and secretary of the Girls "A" Club. She was also a member of the tennis and volleyball teams and the Fellowship of Christian Athletes. Derinda also participated in the Show Choir and the Concert Chorus, where she was vice-president and piano accompanist. She still found time for various community and church activities.

Well-liked and respected by her teachers and fellow students, Derinda was chosen as Most Dependable, Most Likely to Succeed, and Senior Class Favorite. She was also selected for Who's Who Among American High School Students and as TV 31's "Student of the Week".

Through all of her experiences with mathematics, she noticed that "some people are turned off by math because of a bad experience or maybe because they are intimidated by it." She views these people as a challenge. By patiently helping them through a problem, she believes she can make a difference in their mathematical lives. Gwen Snoddy, her sponsor, heartily endorses her selection, "It is not often that a student of this caliber wants to enter the teaching field. I know she will make a wonderful teacher and a valuable addition to the profession."

Mu Alpha Theta is proud to have played a small part in helping Miriam and Derinda along the road to success in mathematics and teaching. Congratulations!

[Bernoulli's Binomial Formula , continued from p.1)
For tossing coins, the probability of success and failure are both $\frac{1}{2}$. To find the probability of rolling [exactly] three 3's in 5 rolls of a die, we proceed the same way. One such sequence is 3, 3, 3, not 3, not 3 whose probability is $(\frac{1}{6})^3(\frac{5}{6})^2$, so

Pr(three 3's in 5 rolls of a die) $=\binom{5}{3}(\frac{1}{6})^3(\frac{5}{6})^2 = .0321$.

In Ars Conjectandi, Bernoulli gave this rule:

If we conduct m + n Bernoulli trials of an experiment where the probability of success is p and the probability of failure is q = 1 – p, then the probability of getting exactly m successes in (m + n) trials is $\binom{m+n}{m} p^m q^n$.

Example 1 Random Randy and Shrewd Sarah take the same 20-question multiple choice test. Each question has five alternatives. Randy makes a random guess on each question and Sarah is always able to eliminate three of the five alternatives and randomly guess among the other two. What is the probability that each student correctly answers a) exactly 60% b) at least 60% of the questions?

Solution: Bernoulli's Rule applies:

60% correct: Randy: $\binom{20}{12}(.2)^{12}(.8)^8 \approx .00009$;

Sarah: $\binom{20}{12}(.5)^{12}(.5)^8 = \binom{20}{12}(.5)^{20} \approx .12$.

To find Pr(correctly answers at least 60% of the questions) we must sum the probability for correctly answering 60%, 65%, 70%, ... , 95%, and 100% of the questions. It is tedious to write down everything, much less carry out the computations. The process is relatively easy on the TI-82 using the [sum] and [seq] keys. These keys are under MATH and OPS on the [LIST] menu. Enter : [sum] [seq]

((20 [n C r] X) [x] (.2[^] X)[x] (.8[^] (20–X)),X,12,20,1)

for Randy and the same for Sarah being sure to change both .2 and .8 to .5. After computing the probability for Randy, press [2nd] [ENTER] to repeat this entry so you can edit it to do the computation for Sarah. The last four entries signify that you are summing the probabilities over the variable, X, as X goes from 12 to 20 in steps of 1. The probabilities are Randy: .0001 and Sarah: .25.

Example 2 Suppose the Brewers and the Giants play in the World Series where the object is to win the best 4-of-7 games. If each team has the same chance of winning, what is the probability the Brewers will win the series in 6 games?

Solution The Brewers must win the 6th game and 3 of the first 5 games, so the probability is $\frac{1}{2}\cdot\binom{5}{3}(\frac{1}{2})^5 = \frac{10}{64} = .15625$.

And now for some exercises:

Exercise 2 Do the problem in Example 2 assuming the probability the Brewers win each game is .6.

Exercise 3 In Example 2, assume the Brewers and Giants have the same chance of winning each game and that the Brewers won the first two games.

a. Find the probability that the Brewers will win the World Series.

b. Find the probability that the Giants will win the World Series.

c. Find the probability that the World Series will take 4 games ? 5 games ? 6 games ? 7 games?

[Many people are surprised at the result of part c]

Exercise 4 A large pool of qualified applicants for a certain job had 55% men and 45% women. Ten people were hired, eight of whom were men. Was there discrimination against women? What is the probability that this would have occured "by chance"?

Exercise 5 "Eight out of ten prefer Zippo cola to Zowie cola" is the sort of claim that some advertisements make. It could be based on only one sample of ten people.

a. Assume Zippo and Zowie are equally good. What is the probability that eight people in a random sample ten people would prefer Zippo?

b. Assume Zippo is preferred by 80% of the people. Now what is the probability that eight people in a random sample ten people would prefer Zippo?

Exercise 6 According to government data, 25% of employed women have never been married.

a. If exactly 10 employed women are selected at random, what is the probability that the indicated number have never been married?

a. exactly two b. two or less c. eight or more

Try to compute the probability that in 500 tosses of a coin, you get 247 heads, or 247 or more heads. You will get an error message because the numbers are too large for the calculator. How to deal with such a problem will be the subject of a future article.

Reference

1. The Mathematical Universe, W. Dunham, Wiley, 1994, Chap. B.

2. The Mathematics of Games and Gambling, E. Packel, New Mathematical Library, Mathematical Association of America, 1981, Chap. 5

3. Introduction to the Practice of Statistics, 2nd ed. Moore, McCabe, Freeman, 1993, Chap. 5