Attend the 1993 Convention in sunny Hawaii, Aug. 4 – 10

The

Mathematical Log

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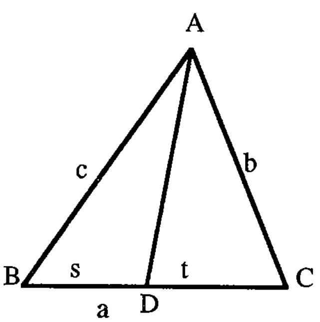
Angle Bisectors and the Steiner-Lehmus Thm

A triangle with two congruent angles is isosceles. Other criteria for an isosceles triangle include two congruent medians, altitudes, or angle bisectors. The first two are usually exercises in a secondary geometry book; the third is renown as the Steiner-Lehmus Theorem. This theorem was sent to the famous Swiss geometer Jacob Steiner in 1840 by C.H. Lehmus with a request for a pure geometrical proof. Over the years there have been many interesting proofs. One "direct" proof involves the formula for the length of the bisector of an angle of a triangle in terms of its sides.

Steiner-Lehmus Theorem
A triangle is isosceles if two of its angle bisectors are congruent.

Problem Let $\overline{AD}$ be the bisector of $\angle A$ in ΔABC. Find AD in terms of a, b, and c.



We consider several ways of solving the problem. All of them require the following lemma.

Lemma Each angle bisector of a triangle divides the opposite sides into segments that are proportional to its adjacent sides; $\frac{s}{t} = \frac{c}{b}$.

Proof: By the Law of Sines,

$$\frac{s}{\sin \frac{1}{2}A} = \frac{c}{\sin D} \quad \text{and} \quad \frac{t}{\sin \frac{1}{2}A} = \frac{b}{\sin D}$$

hence $\frac{s}{t} = \frac{c}{b}$. [Why can we use sin D in both proportions?]

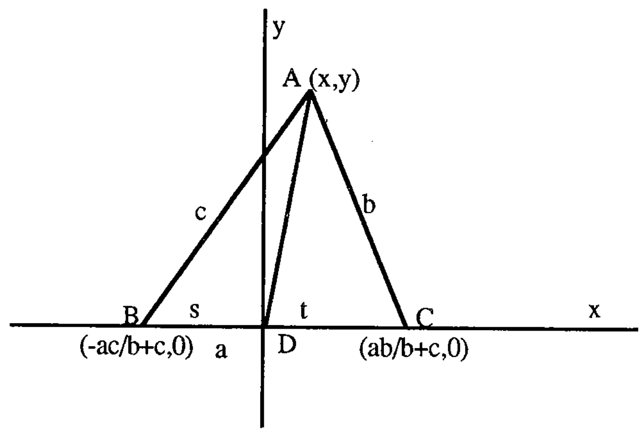
Warning: In the solutions to follow, you will need to perform many algebraic calisthenics to verfy the given equations. Perservere!

Solution 1 [Ali R. Amir-Moez]

Since $s + t = a$ and $\frac{s}{t} = \frac{c}{b}$, we have

$s = \frac{ac}{b+c}$ and $t = \frac{ab}{b+c}$.

We use a coordinate proof choosing a coordinate system as shown with D as the origin and the values for s and t as the x-coordinates of vertices B and C.



By the Distance Formula, (1) $c^2 = (x + \frac{ac}{b+c})^2 + y^2$ and (2) $b^2 = (x - \frac{ab}{b+c})^2 + y^2$. We seek to express $x^2 + y^2$ in terms of a, b, and c.

Subtracting we obtain a linear equation in x that impiies (3)

$$x = \frac{c^2 - b^2}{2a}\left(\frac{(b+c)^2 - a^2}{(b+c)^2}\right) = \frac{(c-b)}{2a(b+c)}[(b+c)^2 - a^2].$$

Now from (1), we get $x^2 + y^2 = c^2 - \frac{2a}{b+c}x - \frac{a^2c^2}{(b+c)^2}$.

Substituting for x from (3) in this equation, we find

$$x^2 + y^2 = \frac{bc}{(b+c)^2}[(b+c)^2 - a^2] = bc\,[1 - (\frac{a}{b+c})^2]$$

There are analogous formulas for the other two bisectors.

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MAθ Bulletin Board

With this issue, we are pleased to begin this new column. Please send any announcements to me or the national office.

• Huneke Award winners are nominated by their students. Think about nominating your sponsor for this award. Your sponsor must have been associated with Mu Alpha Theta for 5 years and have attended or will attend their second national convention. They cannot be a current member of the Governing Council. Write to the national office for nominating forms.

• HAWAII Join the hundreds of members at Brigham Young University for the 23rd national convention in August. Your sponsor already has an announcement. Aloha!

• Governing Council resolution about national conventions:
Only schools with active Mu Alpha Theta chapters may register for MAθ conventions. But an interested school may participate in one (and only one) convention prior to chartering a chapter or reactivating one.

• STATE AND REGIONAL MEETINGS in 1992 - 93.

November 1992: Kansas State Meeting. Hosted by Cowley County Community College. Contact Salem Chaaben, 125 S. 2nd St., Arkansas City, KS 67005

December 4-5: REGION IV Meeting. All chapters in Region IV are invited. Contact Mrs. Mary Emma Bunch or Mrs. Grace Mutz, Farragut HS, 11237 Kingston Pike, Knoxville, TN 37922.

February 1993: Tennessee Regional Meeting. Hosted by Kirby HS, 4080 Kirby Parkway, Memphis, TN 38115. Contact Mrs. Patricia Brownlee.

February 12-13: Texas State Meeting. Held at Corpus Christi State University. Contact Lorraine Dominguez at Moody HS, 1818 Trojan, Corpus Christi, TX 78416.

March 1993: Tennessee State Meeting. Contact J. Michael Bradley, Hickam County HS, 1645 Bulldog Blvd., Centerville, TN 37033.

March 1993: Mississippi State Meeting. Contact Mrs. Claudia Carter, MSMS, P.O.Box W-1627, Columbus, MS 39701.

March 1993: Wisconsin State Meeting. Contact Joe Griesbach, Marquette University HS, 3401 W. Wisconsin Ave., Milwaukee, WI 53208-3842.

March 1993: Louisiana State Meeting. Contact Ms. Barbara Stott, Riverdale HS, 240 Riverdale Ave., Jefferson. LA 70121.

April 1993: South Caroline State Meeting. Contact Mrs. Gloria Allen, South Aiken HS, 232 Pine Log Rd., Aiken, SC 29803-6158.

April 1993: Florida State Meeting. Greenleaf Resort. Contact Mrs. Merita Miller, A.C.Mosely HS, 501 Mosley Dr., Lynn Haven, FL 32444-5609

• New Student Delegate Officers for 1992 - 93. Keep in touch - they want to hear from you.

President: Bill O'Donnell, Parkview HS, 998 Cole Dr. SW, Lilburn, GA 30247-5422

Vice-President: Paul Johnston, White Station HS, 514 S. Perkins Extd., Memphis, TN 38117-3942

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Sgt.-at-Arms: Ed Thompson, Riverdale HS, 240 Riverdale Dr., Jefferson, LA 70121

• FUTURE NATIONAL CONVENTIONS

1994 - New Orleans, Louisiana
1995 - Bowdoin College, Maine
1996, 1997, 1998 ?????

The Mathematical Log

Volume 36 Number 3, October, 1992

The Mathematical Log is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association ofAmerica (MAA) and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to the Editor or to Mu AlphaTheta National Office, 601 Elm Ave., Room 423, Norman, OK 73019.

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1992 $\sqrt{\text{At the Root of It All}}$

Deborah Patonai Phillips, Activities Editor

Huneke and Andree Award Winners

Annually at the national Mu Alpha Theta convention, the organization gives two prestigious awards dealing with education. The Huneke Distinguished Sponsor Award is presented to a MAθ sponsor who has demonstrated a special love and gift for education. The Andree Mathematics Education Award is given either to a graduating senior or to a former MAθ member who plans to become a mathematics teacher. This year is unique in that both recipients come from the same school, the Mississippi School for Mathematics and Science in Columbus, Mississippi.

1992 $\sqrt{\text{At The Root Of It All}}$ congratulates Mrs. Claudia Carter , Huneke Award winner, and her student Doris Tifani Keith, Andree Award winner.

First, let's look at the teacher behind the student. As president of the Mississippi Council of Teachers of Mathematics, Claudia Carter, has been instrumental in making key changes to improve this organization. Known throughout the state as a resource person, she often gives presentations at local, state, and national meetings. She conducts computer training workshops, plans the "Sharing Ideas" summer workshops, and direct staff development sessions. As a Woodrow Wilson master teacher and a member of the Algebra Team, she has been conducting summer institutes throughout the country for several years. In 1989 she received the Mississippi Presidential Award for Excellence in Mathematics Teaching. As one of her co-workers describes her, "If there's a job to be done, Mrs. Carter will take charge, organize the efforts, and see the tasks are completed."

"I am an alien. Please tell me what a function is."

Loved and respected by her students, "she cares for them, treats them fairly, and goes the extra mile to help them achieve." She tries to stimulate their minds [and entertain them] by doing such things as beginning each semester of her precalculus class with, "I am an alien. Please tell me what a function is."

Introduced to MAθ in the late 70's, Claudia has attended all but two of the national conventions since 1980. In 1984 , when the chairperson in New Orleans was unable to fulfill that commitment, Claudia took charge. Making history at the time, the convention attracted over 900 people! Gaining experience from this convention, she has continued to run other local and state MAθ conventions. For the past four years, she has been largely responsible for organizing, planning, and running the Mississippi State Mu Alpha Theta state conventions. According to our Andree winner Tifani Keith, "Mrs. Carter greeted each new pitfall of planning our state convention with calm and reason. She is THE person to praise for the success of our 1992 convention that drew the largest number of participants in our history."

Having Claudia Carter for three semesters in five different classes, Tifani Keith, the winner of the Andree Award, was surely inspired by this remarkable teacher. Attending Duke University, Tifani plans to major in mathematics. After graduation, she wants to teach and pursue her studies in graduate school.

At the Mississippi State School for Mathematics and Science, Tifani was the school's MAθ treasurer. In addition to handling all of the funds, she helped with the scrapbook and found time to tutor students in calculus. As a member of the Interschool Team and Ciphering Team, Tifani helped them to first and second place finishes at the state convention.

At the state level, Tifani was also elected treasurer. She kept the common treasury, attended state board meetings, prepared and distributed a newsletter, and helped organize the state convention. Working for six consecutive hours, she almost single-handedly ran the entire registration process.

Always giving "110%", Tifani loves to be busy and involved. She served as narrator for the annual Math/Science Day dramatic performance and played a supporting role in the first drama production at the school - the original musical Super Math. Reporter for the school newspaper, vice-president of the debate team, captain of the pep squad, co-chairman of the Calendar committee, and a senator in the student government are just a few of her other activities. Often Tifani feels frustrated for not being able to do all that she wants to accomplish. According to Claudia Carter, however, "Tifani accomplishes a great deal with much quality."

Tifani has also received several other honors including being one 2,500 semi-finalists in the U.S. Presidential Scholar Program and attending the National Young

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99 3. Take the number of repeating digits from the original number, 2, and write 9 that many times. Add k zeroes where k is one less than the number of non-

990 repeating digits in the original number. This number is the denominator of the fraction.

Put them over each other and you have the answer: $\frac{5296}{990}$ or $\frac{2648}{495}$.

- Try Joey's method on some other examples. Include some familiar ones such as $\frac{1}{3}$.
- Does his method always work?
- Can you do it in your head?
- Can you explain why his method works?
- Can you find another method?

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From Josh Warren, Central High School, Tuscaloosa, Alabama [Michael Carpenter teacher; Ann Lilly, sponsor]

To prepare for the Mu Alpha Theta competitions, I began to work old Algebra II tests. Since I practiced at home, I usually had to find my own way to solve problems I had never seen before. I often used my computer. One example of such a problem is the bouncing ball problem in which a ball is dropped from a given height and rebounds a given fraction of its original height. The question asks for the total vertical distance traveled.

Unknown to me at the time, this problem involves a geometric sequence. The traditional way to solve it is to add the original height to twice the sum of the geometric series with the first term being the height after the first rebound and the common ratio being the given fraction. Using the numbers I saw in one problem, I wrote a GBASIC program that continually "bounced" the ball until the difference between two consecutive heights was rounded off to 0 by the computer. The given fraction was $\frac{4}{5}$, and I noticed that the answer was a multiple of 9. Futhermore, the number multiplied by 9 was the original height.

Noticing that 9 was the sum of the numerator and the denominator of the fraction, I wrote a program to test each fraction between 0 and 1 with a denominator of 10 or less and a unit height. I first noticed that the answer for fractions with a numerator one less than the denominator was the sum of the two. I then compared $\frac{1}{2}$ with $\frac{2}{4}$ and $\frac{3}{6}$. I noticed that though they had the same answer, that answer was equal to the sum of 1 and 2, half the sum of 2 and 4, and a third of the sum of 3 and 6. I associated my observance with the difference and came up with the equation $X = \frac{H(D+N)}{D-N}$ where X is the distance traveled, D is the denominator of the fraction, N is the numerator, and H is the original height. I used my program to compare the results of my formula with the manual process of adding each individual height to test the formula's validity for different heights and fractions. The formula was true in every case.

I have not yet tried to explain why this formula works. I have never seen it nor known anyone who knows it.

- Try Josh's method on some other examples.
- Does his method always work?
- Can you help Josh to explain why his method works?

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($\sqrt[1992]{\text{At The Root Of It All}}$ continued from page 3)

Leaders Conference, a one-week session of government in Washington D.C. Serving as co-captain, she helped her team place in the top 20 out of 60 teams in the World Finals of the Odyssey of the Mind Competition. She has been listed in <u>Who's Who Among American High School Students</u> three times and placed third in a pumpkin carving contest once.

Undoubtedly inspired by her teacher, Claudia Carter, Tifani has often thought teaching as a career. She has always loved mathematics, believing that it "is the only subject that does not discriminate ... one of two universal languages." Through teaching, she hopes to share this love with others, because "the best thing you can do with an education is to share it so that it multiplies."

Articulate, creative , enthusiastic, Tifani "is one of those are individuals who has the talent of bringing sunshine into a world full of rain and an air of peace into the turbulence of a storm." Mrs. Carter remarks: "After having worked with Mu Alpha Theta for many years, I am extremely pleased that I FINALLY have a student that I can nominate for this award."

Claudia and Tifani have received more than just an award - each has inspired the other. The world of education needs both of you! Congratulations!

CONTEST CORNER

This month's column contains the Power Question from the 1992 ARML Contest. It was composed by Gil Kessler and Larry Zimmerman and is an interesting combination of number theory and analytic geometry. The solution will appear in the next issue.

Throughout this problem, the points A (a,a^2), B (b,b^2), C (c,c^2), and D (d,d^2) represent distinct lattice points on the parabola $y = x^2$.

I. Let the area of ΔABC be K. It can be shown that $K = \frac{1}{2}\left|(a-b)(b-c)(a-c)\right|$.

1. Show that K must be an integer.
2. Show that K = 3 is the only possible prime value for K.
3. Show that K cannot be the square of a prime.
4. Show that the area of quadrilateral ABCD cannnot be 8.

II. It can be shown that the slope of line AB is a + b.

1. A line passes through the point (3,5) and through two lattice points on $y = x^2$. Compute the coordinates of these two points being sure to find all possible pairs of such points.

2. A line passes through the point (2,4) and through three other lattice points on the "double parabola" $y^2 = x^4$. Compute the coordinates of these three points being sure to find all possible triplets of such points.

III. Consider quadrilateral ABCD. (Remember the slope of line AB, for example, is a + b.)

1. Let the vertices be labeled (alphabetically) in a counterclockwise direction. Show that $\tan A = \frac{d-b}{1+(a+b)(a+d)}$.

2. A quadrilateral is "cyclic" if all four of its vertices lie on the same circle. Show that: If quadrilateral ABCD is cyclic, then a + b + c + d + 0; AND If a + b + c + d + 0, then quadrilateral ABCD is cyclic.

3. Use the previous result to show that: If a circle intersects the graph of $y = x^2$ in four points, and three of them are lattice points, then the fourth point must also be a lattice point.

dia Log ue

with Log Editor Tom Butts

This year's convention in Princeton was a great success. Congratulations to the many award and contest winners. Special thanks to the host chapters who organized the convention, to the contest writers, and to all the others who made the convention such an enjoyable and worthwhile experience.

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Thanks also for the 20 entries that were submitted to the MAΘ Essay Contest. There were two poems, several essays, a radio script, an art piece, and papers of varying length. A special thank you to all the teacher-sponsors who worked with those who submitted entries. Although the judging has not been completed, we present excerpts from two of the short papers. You are encouraged to send your comments on either one of them as well as questions, comments, suggestions on what type of articles and features to include in the Log, a question about any area of mathematics - a person, a concept, a problem,... a comment about any of the other articles , any contributions for the *Logniappe* colummn, or anything else on your mind. Send it to me at he address on page 2.

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From Joey Epstein of Bellflower High School in Bellflower, California [James Fouquette, teacher]

A Simpler Method of Determining a Fraction Form of a Repeating Decimal

When asked for the fraction equivalent of a repeating decimal like $5.3\overline{49}$ or 5.3494949494 ... you don't need to use the usual method of finding

$$100x = 534.9\overline{49}$$
$$x = 5.3\overline{49}$$
$$99x = 529.6$$

and dividing to get $\frac{5296}{990}$ or $\frac{2648}{495}$.

There's a shorter method - one that you can probably do in your head!

First ...

5349	1) Take the digits you see
5349 - 53 5296	2) Subtract the number formed by the non-repeating digits in the original number from the number formed in step 1.

(continued on page 4)

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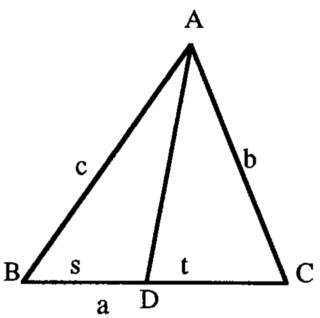
We can test the plausibility of this formula be examining some special cases. If the triangle degenerates, collapses into a line segment, then $a = b + c$ and the formula yields $AD = 0$. If the triangle is equilateral, then $a = b = c$ and the formula yields $AD^2 = a^2[1 - (\frac{1}{2})^2]$ or $AD = \frac{\sqrt{3}}{2}a$ as we expect.

Solution 2

Again, since $s + t = a$ and $\frac{s}{t} = \frac{c}{b}$, we have $s = \frac{ac}{b+c}$ and $t = \frac{ab}{b+c}$. Let $d = AD$.



Applying the Law of Cosines to ΔABC and ΔABD yields
$b^2 = a^2 + c^2 - 2ac \cos B$ and $d^2 = s^2 + c^2 - 2sc \cos B$.
Eliminating $\cos B$ and solving for d^2 yields
$d^2 = bc\,[1 - (\frac{a}{b+c})^2]$ as before.
Alternatively one could apply the Law of cosines to ΔABD and ΔACD using $\frac{1}{2}\angle A$ in each case.

Solution 3

Let $AD = d$ and $\angle BAD = \angle DAC = x$. Using the problem solving strategy of expressing the same quantity in two different ways, we express the area of ΔABC as
$\frac{1}{2} bc \sin 2x = \frac{1}{2} cd \sin x + \frac{1}{2} db \sin x$.
Since $\sin 2x = 2 \sin x \cos x$, we have $d = \frac{2bc \cos x}{b + c}$ or
$d^2 = \left(\frac{2bc \cos x}{b + c}\right)^2$.
To find $\cos^2 x$, we apply the Law of Cosines to ΔABC to obtain $a^2 = b^2 + c^2 - 2bc \cos 2x$ and then use the identity $\cos 2x = 2\cos^2 x - 1$.

To prove the Steiner-Lehmus Theorem, we set (for example)
$ac\,[1 - (\frac{b}{a+c})^2] - ab\,[1 - (\frac{c}{a+b})^2] = 0$ and show that the left side has a factor of $(b - c)$. Thus $b = c$ and ΔABC is isosceles.

There are many problems, of varying degrees of difficulty, that involve angle bisectors. Some of them are:

1. Does the result of the Lemma hold for the bisector of the exterior angle of a triangle? Justify your answer.

2. Prove: In a trapezoid whose diagonals are the bisectors of the angles of one of the bases, three of the sides are congruent.

3. Find the lengths of the bisectors of the acute angles in a right triangle whose legs are 18 and 24.

4. Prove: In a right triangle the bisector of the right angle also bisects the angle between the median and the altitude from the right angle.

5. Find the length of the bisector of the right triangle whose legs are a and b. [Try to find a way other than applying the formula above.]

6. Prove: In any triangle, the angle bisector either coincides with the median and altitude drawn from the same vertex or lies between them.

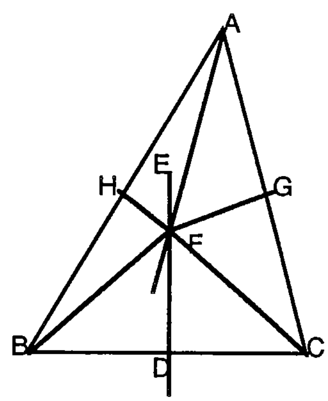
7. [Geometry from Multiple Perspectives, NCTM Addenda Series for grades 9 - 12]
If you were to draw the bisectors of the angles of a square, they would meet in a single point since they are the diagonals. But what happens for other quadrilaterals? Draw the bisectors of the four angles of the given quadrilateral and make a list of the properties of the four points of intersection. [e.g. are they the vertices of any special type of quadrilateral?]

a. rectangle
b. rhombus
c. parallelogram
d. kite
e. isosceles trapezoid
f. trapezoid
g. arbitrary quadrilateral

8. In a parallelogram with sides a and b and acute angle a, find the area of quadrilateral whose vertices are the points of intersection of the four angle bisectors. [see #7]

The marvelous absurdity: Every triangle is isosceles, also involves bisectors.

Let ABC be any triangle. Bisect BC at D and from D draw DE perpendicular to BC. Bisect $\angle BAC$.
1. If the bisector does not meet DE, it is perpendicular to BC [being parallel to DE] and hence ΔABC is isosceles.
2. Otherwise it meets DE at F. Join FB, FC and from F draw FG and FH at right angles to AC and AB. Then $\Delta AFG \cong \Delta AFH$ by HA. Thus $AH = AG$ and $FH = FG$. Now $\Delta BDF \cong \Delta CDF$ by SAS and $FB = FC$. So $\Delta FHB \cong \Delta FGC$ by HL and $HB = GC$. Thus $AB = AC$ and ΔABC is isosceles.



9. Can you find the flaw in this "proof"?