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1, 2, 3, 4, 5,.....Three Counting Techniques

(1) How many triangles are determined by 10 points, no three of which are collinear?

(2) How many 3-digit numbers have three different digits in decreasing order?

Both problems are examples of combinatorial problems or counting problems in which the basic question is "How many .... ?" or "In how many ways ... ?" Combinatorics is an area of discrete mathematics, the subject of the 1991 NCTM Yearbook.

Both problems have the same answer: C(10,3) - the number of 3-element subsets in a set of 10 elements. C(10,3) is the fourth number in the tenth [or eleventh] row of Pascal's Triangle and is the coefficient of x^3y^7 [and x^7y^3] in the binomial expansion $(x + y)^{10}$. It equals $\frac{10!}{3!7!} = 120$ and can be computed on many (graphing) calculators. It is also denoted by ${}_{10}C_3$ or $\binom{10}{3}$.

To see that the two problems have the given solution, you can think: (1) every subset of 3 of the 10 given points are the vertices of one triangle, and (2) every subset of 3 of the ten digits {0,1,2,3,4,5,6,7,8,9} corresponds to one 3-digit number with three different digits in decreasing order.

In this article, we examine three other counting techniques: some special combinations, the Inclusion-Exclusion Principle, and the use of generating polynomials.

some special combinations

Example 1 How many positive integer solutions are there to the equation x + y + z + w = 12? One such solution is x = 2, y = 5, z = 1, w = 4.

Solution Consider 12 1's separated by 11 spaces: 1s1s1s1s1s1s1s1s1s1s1s1. To find a solution, we must choose 3 of the 11 spaces. Suppose, for example, we choose the third, sixth and tenth spaces [and omit the others]: 1s1s1[s]1s1s1[s]1s1s1s1[s]1s1 or 111s111s1111s11. This choice corresponds to the solution: x = 3, y = 3, z = 4, w = 2. There are C(11,3) = 165 such choices and hence there are 165 positive-integer solutions to the equation.

Example 2 The cafeteria sells three different flavors of ice cream, say vanilla, chocolate, and strawberry. In how many ways can you order 6 ice cream cones?

Solution Let the three flavors be F_1 [vanilla], F_2 [chocolate], F_3 [strawberry]. Then any order of 6 cones can be described by listing the flavors in increasing order. Thus $F_1F_1F_1F_2F_2F_3$ would mean an order of 3 vanilla, 2 chocolate and 1 strawberry while $F_2F_2F_3F_3F_3F_3$ would mean an order of 2 chocolate and 4 strawberry. Let us draw vertical lines to separate the flavors: $F_1F_1F_1 | F_2F_2 | F_3$ and $| F_2F_2 | F_3F_3F_3F_3$. In each case, the result is a sequence of 8 symbols - 6 F's and 2 vertical lines.

Conversely we could start with a row of 8 stars: * * * * * * * *. Pick any two of the stars and draw bars through them: * * ∤ * * * ∤ *. This scheme detemines a choice of 6 cones: 2 vanilla, 3 chocolate, and 1 strawberry. The scheme ∤ ∤ * * * * * * would mean 6 strawberry cones.

It follows that the number of ways to order 6 cones is C(8,2) = C(8.6) = 28: Choose 2 bars from 8 objects or, equivalently, choose 6 stars from 8 objects.

Problem 1 a) How many different assortments of a dozen doughnuts are possible if there are 5 varieties of doughnuts from which to choose? b) How many assortments are there if you want to include at least one of each type of doughnut?

Problem 2 How many a) non-negative integer b) positive integer solutions are there to the equation x + y + z = 17?

the Inclusion-Exclusion Principle

Example 3 How many factors of 900 are multiples of 3 or 5?

Solution Let $|X|$ denote the number of elements in set X. If A = {multiples of 3 ≤ 900} and B = {multiples of 5 ≤ 900}, then the desired number of factors is $|A \cup B|$. But $|A \cup B| = |A| + |B| - |A \cap B|$, so there are 300 + 180 – 60 = 420 such factors since $A \cap B$ = {multiples of 15 ≤ 900}.

(continued on page 5)

See story on page 3

Remember: MAθ Essay Contest

You are strongly encouraged to enter the MAθ Essay Contest. The type of entry you submit is limited only by your imagination, but several possibilities are:

- a paper containing an original discovery
- your discussion and solution of an interesting problem
- an expository paper organized in a thought-provoking manner
- a "short subject" such as a poem, a cartoon, etc.

You are also encouraged to submit photos of models, posters [e.g. used in Science Fair], etc. to supplement your entry.

This year's deadline is June 15, 1992. Submissions should be sent to the Editor at the address on this page. Each entry must be signed by you and your sponsor to indicate that the work is original. Winners, as determined by a panel of judges, could receive a cash prize of up to $1,000. The number and amount of the prizes is determined by the judges. Winning entries will be published in the Log and/or in the next issue of Mathematical Buds. Some details are awaiting final approval by the governing board.

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Logniappe

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WORDS IN COMMON - Answers

1. line 2. triangle 3. plane 4. term 5. ray 6. root 7. square 8. disc 9. cube 10. space 11. sign 12. circle

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From the list below, choose the term that describes each object. Actually you might cover the list and use it only if you need it.

1. That which Noah built.
2. What a bloodhound does in chasing a woman.
3. An appropriate title for a knight named Koll.
4. A tall coffee pot perking.
5. When one does when it rains.
6. What a boy does on the lake when the motor won't run.
7. What the captain said when the cannonball hit his ship.
8. What one does to trees when they are in the way.
9. Can George Washington turn into a country?
10. A small dog sitting in a refrigerator.

PERPENDICULAR	INSCRIBE
DECAGON	CENTER
COSECANT	COINCIDE
HYPOTENUSE	CIRCLE
ARC	HERO

The Mathematical Log

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1992 √At the Root of It All

Deborah Patonai Phillips, Activities Editor

Friendships, both new and old, are hallmarks of Mu Alpha Theta. By planning and performing various activities and by practicing for competitions, members, as well as sponsors, grow closer to each other. In state, regional, and national conventions, more friendships are formed. Some of them last for many years, perhaps a lifetime. A perfect place to renew old friendships and to make new ones occured at the first Region IV Convention. √At the Root Of It All is proud to recall the congenial atmosphere of that meeting.

In early December, Mary Rhein and Nancy McDaniel, MAθ sponsors from Lakota High School, hosted their first regional convention. Held at Lakota High School, in West Chester, Ohio, the convention brought together over 160 students and sponsors from 12 schools in 5 states: Ohio, Tennessee, Maine, Kentucky, and Vermont.

This convention introduced the Leonardo da Vinci Competition in which teams, including one composed of sponsors, used only colored tissue paper, toothpicks, and glue to build an airplane.

The two-day convention featured a variety of activities and tests. Students competed in an interschool test, in a written competition, in ciphering rounds, and in group relays. For some tests, students were grouped with those from schools other than their own. Friendships were blossoming! This convention introduced the Leonardo da Vinci Competition in which teams were composed of students from different schools and one group of talented sponsors [including yours truly!]. Using only colored tissue paper, toothpicks, and glue, each team was to build an airplane. Each plane was judged on its appearance, design, construction, hang time, and distance covered. This event was certainly great fun!

While everyone attending the convention was a winner, some schools did display their mathematical prowess. Winners in the Interschool Test were: Farragut, TN; Beavercreek, OH; and Paul Blazer, KY. In the Relay, the top schools included Beavercreek, Oak Ridge, TN; and Farragut. Top finishers in the Math Bowl were Beavercreek [in a triple tie-breaker], Paul Blazer, and Centerville, OH. Receiving a trophy,$50, and a scholarship to the national convention at Princeton was the top individual winner: Hareendra Yalamanchili from Paul Blazer in Ashland, KY. Other top scorers included: Mark Roh, Beavercreek; David Blau, Oak Ridge; Dan Schepler, Beavercreek; and George Chong, Beavercreek. Congratulations to all.

This first regional convention was a huge success that required a great deal of time and effort. Mary Rhein achieved her goal of "encouraging more schools in Region IV to participate in the national convention by giving them a taste of what it would be like." She appreciated the efforts of parents, staff, and some colleagues from local universities who wrote most of the questions. ATTENTION Region IV sponsors: Where will next year's conference be held?

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One of the five schools participating in a convention for the first time is located in Bluffton, OH. With encouragement from Mary Rhein, the "math club", with sponsor Duane Bollenbacker, joined MAθ this fall with 15 students becoming charter members of the chapter. One of their first endeavors was to sponsor a problem solving contest involving students, faculty, and administrators. They also offered a coat check service at athletic events and music activities. Several hundred students, parents, and community members enjoyed "Mathematics in the Soap Bubble" presented by retired Bluffton College professor, Dr. L. Shetler.

The highlight of the year was a "Math Fun Night of Problem Solving" for students from neighboring schools.

The highlight of the year was a "Math Fun Night of Problem Solving" for students from neighboring schools. As they arrived, all participants were assigned to teams with students from schools other than their own. "The purpose was NOT to compete school versus school, but to enjoy the problems through discussion with students from other schools". Students and advisors still talk about the fun they had and the friendships they made at this event. Plans are underway for a repeat performance next year.

At MAθ conventions, a friendly rivalry exists among schools. The emphasis, however, should be on the "friend" in friendly. The first regional conference in Region IV made friends of us all!

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In the photo on page 2, Mu Alpha Theta students at Pecos High School in Pecos, Texas decorated the tree on the right. On display at the West of the Pecos Museum, this tree was one of fourteen donated for future planting at area schools. Following the theme of "Christmas Around the World", these MAθ members chose Lithuania as their country because the traditional Christmas decorations there were geometric figures woven from straw. Instead of weaving ornaments, the members made their decorations form paper and their garlands from foam packing material shaped like the integral symbol.

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Please send news and photos from your chapter to the address on page 2. I'll publish them in future issues of the Log.

A Millenium of Mathematics (and a little more)

-300 Euclid - Euclid's Elements and Euclidean Algorithm [Write a computer program for the Euclidean Algorithm*]

-250 Eratosthenes - sieve for prime numbers*; measure the circumference of the Earth*

-225 Apollonius - conic sections; the circle of Apollonius - If A and B are fixed points, then the locus of a point P such that $\frac{AP}{BP} = k$ is a circle if $k \neq 1$. Verify using analytic geometry. What happens if $k = 1$?*

-225 Archimedes - greatest mathematician of antiquity - volume of the sphere [Verify his result that the volume of the sphere is $\frac{2}{3}$ that of the circumscribed cylinder and the surface area is $\frac{2}{3}$ that of the total surface area of the circumscribed cylinder*]- method of integration*

36 largest two-digit whole number that is divisible by the sum of its digits*

40 Biblical expression for a long period of time, e.g. 40 days in the wilderness

41 $x^2 + x + 41$ is prime for $x = 0, 1, 2, \ldots, 39$

50 smallest whole number that can be written as the sum of two squares in two different ways; special investigation unit of the Hawaii state government

60 base used by Babylonians for mathematical and astronomical work

101 number of dalmatians of Dodie Smith later immortalized by Disney

109 number of John F. Kennedy's PT boat

150 Ptolemy - table of chords to approximate values of the trigonometric ratios - Use Ptolemy's Theorem [look up] to prove that if a and b are two chords of a unit circle, then the sum, s, is

$s = \frac{a}{2}\sqrt{4 - b^2} + \frac{b}{2}\sqrt{4 - a^2}$. To what familiar trigonometric addition formula does this formula correspond?*

250 Diophantus - Diophantine equation; His epitaph: "Diophantus passed $\frac{1}{6}$ of his life in childhood; $\frac{1}{12}$ in youth; and $\frac{1}{7}$ more as a bachelor. Five years after his marriage was born a son who died 4 years before his father at $\frac{1}{2}$ his father's [final] age." How old was Diophantus when he died?*

300 Pappus - Pappus' theorem; Prove Pappus' extension of the Pythagorean Theorem: Let ABC be any triangle, and ABDE, ACFG, any parallelograms described externally on AB and AC. Let DE and FG meet in H and draw BL and CM equal and parallel to HA. Then area BCML = area ABDE + area ACFG.*

360 number of degrees in a circle

371 one of four whole numbers equal to the sum of the cubes of its digits - find the others*

451 alleged temperature, in degrees Fahrenheit, at which books burn

500 length of race, in miles, held annually on Memorial Day at the Indianapolis Motor Speedway

567 $567^2 = 321489$: One of two such equations that use each of the nine nonzero digits once - find the other one*

628 Brahmagupta - area of quadrilaterals - Prove $K^2 = (s-a)(s-b)(s-c)(s-d)$ for the area K of a cyclic quadrilateral with sides a,b,c,d and semiperimeter s.*
[see Log, Dec. 1990]

641 $2^{2^5} + 1$ is divisible by 641 - the first counterexample to Fermat's conjecture that $2^{2^n} + 1$ is prime

755 number of home runs hit by Henry Aaron

775 Alcuin of York, compiler of Problems for the Quickening of the Mind. Try this one: If 100 bushels of corn be distributed among 100 people so that each man receives 3 bushels, each woman 2 bushels, and each child $\frac{1}{2}$ bushel, how many men, women, and children were there?*

820 Mohammed ibn Musa al-Khowarizmi - treatise on algebra

840 number less than 1000 with the greatest possible number of factors - why?*

880 number of 4x4 magic squares [provided that all reflections and rotations of the same square are counted as one]. Find one 4x4 magic square.*

945 smallest odd abundant number - why?*

1089 $9 \times 1089 = 9801$ - find the other such 'palindromic' equation involving a 4-digit number and one of its digits.*

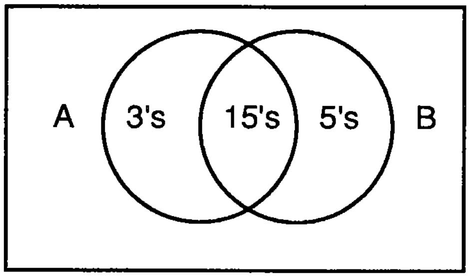
If a 3-digit number is subtrtacted from its reversal and that answer is added to its reversal, what is [are] the possible result(s)? Ex. 623 – 236 = 297; 297 + 792 = 1089.*

1095 the first Crusade

1100 Omar Khayyam - solution of cubic equation: Show that the cubic equation $ax^3 + bx + c = 0$ can be solved by graphing $y = x^3$ and $ay + bx + c = 0$ [How?]. Use the equation $x^3 + 6x - 15$ as an example.*

You are encouraged to look up the topics and/or solve the problems marked with an * .

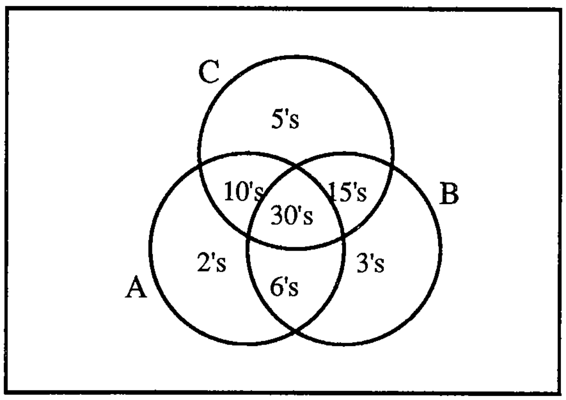
(continued from page 1)



The Inclusion-Exclusion Principle is a generalization of this principle to three or more sets. Let us first consider an example with three sets.

Example 4 How many factors of 900 are multiples of 2, 3 or 5?

Solution If A = {multiples of 2 ≤ 900}, B = {multiples of 3 ≤ 900} and C = {multiples of 5 ≤ 900}, then the desired number of factors is $|A \cup B \cup C|$.



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- (|A \cap B| + |A \cap C| + |B \cap C|)$$
$$+ (|A \cap B \cap C|).$$

To show the validity of this formula, we consider an arbitrary element in each of the three types of sets. Consider an element e that is in only one of the three sets. If e = 14, for example, then e is counted once in the term $|A|$. Next consider an element f that is in exactly two of the sets. If f = 18, for example, then f is counted positively in $|A|$ and $|B|$ and negatively in $|A \cap B|$. Hence it is counted 1 + 1 - 1 = 1 time in the formula [as it should be]. Finally if g is an element that is in all three sets, g = 60 for example, then g is counted once in each of the seven terms of the formula and is therefore counted 1 + 1 + 1 - (1 + 1 + 1) + 1 = 1 time.

Example 5 Let X = {5,6,7,8,9,10} and Y = {1,2,3,4}.
a) How many functions map X to Y?
a) How many functions map X onto Y?

Solution a) For each of the 6 elements in the domain X, there are 4 choices for its image in Y. Thus there are 4x4x4x4x4x4 or 4^6 functions.

b) "onto" means that each of the four elememts 1, 2, 3, and 4 in Y must occur as an image for some element in X. From the total number of functions, 4^6, we subtract those do not contain all of the elements 1, 2, 3, 4 as images. Let

A_1 = {functions that do not have 1 as an image},
A_2 = {functions that do not have 2 as an image},
A_3 = {functions that do not have 3 as an image},
A_4 = {functions that do not have 4 as an image}.

The number to be subtracted, then, is $|A_1 \cup A_2 \cup A_3 \cup A_4|$.

We use the Inclusion-Exclusion Principle, for 4 sets, to compute this number. Write out this formula for yourself before you proceed.

Since A_1 contains those functions that do not have 1 as an image, there are only 3 choices [for an image] for each of the six elements in X. Thus $|A_1| = 3^6$. Similarly $|A_2| = |A_3| = |A_4| = 3^6$.

Similarly since $A_1 \cap A_2$ contains those functions that do not have 1 or 2 as an image, there are only 2 choices [for an image] for each of the six elements in X. Thus $|A_1 \cap A_2| = 2^6$. The same is true for each of the other C(4,2) – 1 = 5 intersections of two sets.

In the same way $A_1 \cap A_2 \cap A_3$ contains those functions that do not have 1, 2, or 3 as an image; i.e. the constant function f(x) = 4. The same is true for each of the other C(4,3) – 1 = 3 intersections of three sets i.e. the other three constant functions f(x) = 1, f(x) = 2, and f(x) = 3.

Finally $A_1 \cap A_2 \cap A_3 \cap A_4$ is empty.

Putting all this together, we have $|A_1 \cup A_2 \cup A_3 \cup A_4| = C(4,1)\cdot 3^6 - C(4,2)\cdot 2^6 + C(4,3)\cdot 1^6 - C(4,4)\cdot 0 = 4\cdot 3^6 - 6\cdot 2^6 + 4\cdot 1 - 0 = 2536$. The number of "onto" functions, then, is $4^6 - 2536 = 1560$.

Problem 3 a) How many factors of 1260 are multiples of 2, 3 ,5, or 7?
a) How many positive integers ≤ 1260 are are relatively to 1260?

Problem 4 a) A bridge hand consists of 13 cards chosen from a deck of 52 cards. How many different bridge hands contain a [at least one] void suit (that is, no cards in that suit)? [Ans. 32,427,298,180.]
b) What fraction of all possible bridge hands contain a [at least one] void suit?

(continued on page 6)