Attend the 1991 Convention in Huntsville, Alabama

The Mathematical Log

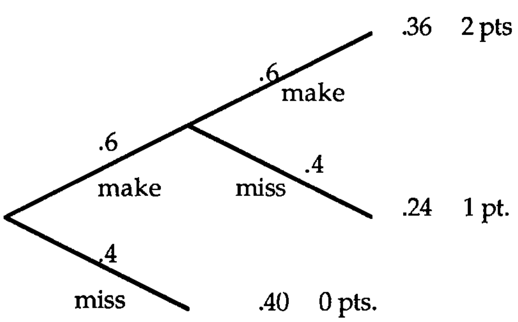
Volume 35 Number 1, February 1991

Great Expectations

Consider the problem:

Problem Charlie Dickens is a 60% free throw shooter. He has a "one-and-one" situation [usually called a trip to the line] , i.e. if he makes the first free throw, he gets to attempt a second one. In this trip, how many points is he most likely to score: 0, 1, or 2 ?

Many people will instinctively respond "1", but let's investigate using a probability tree diagram.



Thus 0 is most likely, followed by 2 and then 1. Over the long term, Charlie's average number of points per trip is, however, close to 1. If Charlie made 100 trips, for example, he could be expected to score 0 points on 40 trips, 1 points on 24 trips, and 2 points on 36 trips for an average of $\frac{(40)0 + (24)1 + (36)2}{100} = .96$ points per trip. Most people's instinctive reasoning concerns the long -term average instead of the most likely outcome on any given trial.

We can rewrite this average as $(.40)0 + (.24)1 + (.36)2 = .96$. In general, then, if [pairwise disjoint] events $E_1, E_2, \ldots, E_n$ occur with probabilities $p_1, p_2, \ldots, p_n$ and payoffs $r_1, r_2, \ldots, r_n$, respectively, then the long-term average or **expected value** is given by $X = p_1 r_1 + p_2 r_2 + \ldots + p_n r_n$. In the problem above, $p_2 = .24$ and $r_2 = 1$ (point). As in this problem, the expected value need not equal any of the possible payoffs. A game is **fair** if its expected value is 0. Most gambling games are unfair since their expectation is negative. Perhaps the most unfair game is a state lottery. [see the Log , Feb. 1985, and Tall Timbers, April 1988]

Example In the PICK 3 lottery game, you select a 3-digit number from 000 – 999. If your number matches the winning number, you win $500. If the first two digits or the last two digits match those of the winning number, you win $50. If it costs $1 to play the game, what is your expected loss?

Solution:

Event	Probability	Payoff	Expectation
3-digit match	.001	$500	$ .50
2-digit match(F)	.009	$50	$ .045
2-digit match(L)	.009	$50	$ .045

Most lotteries pay only the highest amount on any winning ticket, so there are 9 tickets out of 1000 that match the first two digits, but not the third. The expected value is .001(500) + .045(50) + .045(50) = .59. Since the ticket costs $1, you can expect to lose $ .41 on each play of this game.

Two gaming problems are frequently cited as the beginning of the science of probability. In the 1650's the Chevallier deMere noticed that made considerable money by betting that he could throw at least one six in 4 rolls of a die. He reasoned that he should also win by betting that he could throw one double-six in 24 rolls of a pair of dice. What is his expectation in the first case assuming he wins or loses $100 on each set of 4 rolls?

Continued on page 6

"As Close As Possible" - An Interesting Problem Variation

It is well-known that the Diaphantine equation $x^2 + y^2 = z^2$ has many [integer] solutions. If we change the right side of this equation "by 1", do the resulting equations have solutions?

Problem 1 For each equation, either find as many [integer] solutions as you can or prove that none exist.

a. $x^2 + y^2 = z^2 + 1$ b. $x^2 + y^2 = z^2 - 1$

We constructed a variation of a problem by changing a condition of the problem. In this case, we tried to "come as close as possible" to the original condition. The new problems that result using this technique can be celebrated, interesting, routine, easy, difficult, or unsolvable. In the examples we will now investigate, severalof these adjectives can be applied. Many of them are ideal for computer exploration.

Problem 2 The equation $x^3 + y^3 = z^3$ has no integer solutions. "How close can you come?" That is, find the minimum value of $| x^3 + y^3 - z^3 |$ for integers x,y,z. Can you prove the value you obtained is the minumum? How many times does it occur?

The sum of the divisors [factors] of a natural number n is denoted by $\sigma(n)$. Thus $\sigma(17) = 1 + 17 = 18$ and $\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$. A number n is **perfect** if $\sigma(n) = 2n$. The first three perfect numbers are 6, 28, and 496.

Problem 3 A number n is **almost perfect** if $\sigma(n) = 2n - 1$. Are there any almost perfect numbers?

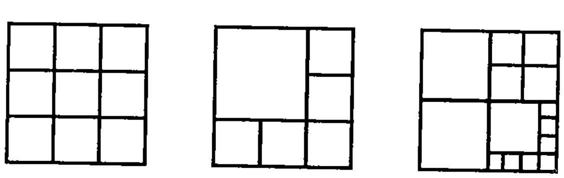
Problem 4 A number n is **quasi-perfect** if $\sigma(n) = 2n + 1$. Are there any quasi-perfect numbers?

Problem 5 Let $S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$.

a. Prove S_n is never an integer.

b. How close can S_n be to an integer? That is, find the minimum value for $n > 1$ of $| S_n - [S_n + .5] |$ where [x] is the greatest integer function. [You might investigate the floor and ceiling functions.]

There are many ways to partition a square into smaller squares.



If no two of the smaller squares are congruent, then the original square is **perfect.**

Problem 6 Can you find a perfect square?

The variation of this problem is easier and may be worthwhile to attack first.

Problem 7 Find an n+1 by n rectangle that can be partitioned into squares with integer sides. No two of the squares are congruent. [Hint: $30 < n < 36$]

The Editor would welcome correspondence on any of these problems or original problems constructed using this technique.

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The average percent of the money wagered in a state lottery that is returned as prizes varies from 59.5% in Massachusetts to 43.6% in Arizona.

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Last issue answers: 1. denominator 2. addition 3. subscript 4. rhombus 5. circumference.

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The Mathematical Log
Volume 35 Number 1, February 1991

The Mathematical Log is the official publication of Mu AlphaTheta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association ofAmerica (MAA)and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to the Editor: Thomas R. Butts,University of Texas at Dallas,P.O. Box 830688 Richardson,TX 75083-0688,or to Mu AlphaTheta National Office, 601 Elm Ave., Room 423, Norman, OK 73019.

$\sqrt[1991]{\text{At the Root of It All}}$

Deborah Patonai Phillips, Activities Editor

Salute to Huneke and Kalin Winners

"She is what every student envisions a teacher ought to be: intelligent, caring, motivating, and principled." With these words, MAΘ president Suresh Karne described Grace R. Mutz, this year's recipient of the Huneke Distinguished sponsor Award. Also honored in this column is Alexander J. Hartemink, winner of the 1990 Kalin Award for an outstanding student.

"She is what every student envisions a teacher ought to be: intelligent, caring, motivating, and principled."

In 1979 Grace Mutz helped to charter the MAΘ chapter at Farragut High School in Knoxville, Tennessee. Serving as co-sponsor since that time, Grace has encouraged her members to be active at local, state, regional, and national levels. In the short history of the chapter, Farragut members have been elected as the regional governor five times, as a state officer four times, and as a national officer twice. Introduced to mathematical competitions at various conventions, she has helped her teams to become extremely competitive by holding practices before and after school, using practice exams, and teaching various "tricks" for doing certain types of problems quickly. Testifying to her success are the many awards and trophies that now go to Farragut each year.

Under her direction the club keeps very active. They run a student tutoring program giving members "an opportunity to share their knowledge and understanding of mathematics." The number of students participating in the Math Bowl at Farragut has increased from 150 in 1981 to over 1000 in each of the last few years. She has initiated a point system to encourage her members to be active. They receive points for participation in competitions, tutoring, meetings, and fundraisers with awards going to those earning the most points.

In 1988 Grace took on the momentous task of chairperson of the national MAΘ convention at the University of Tennessee in Knoxville. She spent countless hours over a two-year period planning and organizing the many activities and arrangements for the convention. It truly was a GREAT convention! As her friend and co-sponsor Mary Emma Bunch says, "this was another example of her love for MAΘ, the students, and mathematics. She is always willing to go the extra mile to help."

Kalin Winner

This year's recipient of the Kalin Award is Alexander Hartemink, a graduate of Fort Myers High School in Fort Myers, Florida. Active in MAΘ since eighth grade, he served as treasurer, vice-president and president of the club during his high school years. Not many students can say they have attended FIVE national conventions!

Alex worked many hours with the various math teams, his favorite being the calculus team. As captain, he worked with five other students, "drilling them on concepts, preparing materials for study, and inspiring them to do their best." He has been quite successful in various competitions including earning first place in Calculus, Functions, and Theory Topics tests at the Chicago national, and first place overall Alpha Division Champion at the Tampa national. He also placed first in the Florida Olympiad, in the Florida Math League, and in the national Math League. He received a score of 139 on the AHSME placing first in Florida and seventh nationwide. He has also been very active in other school activities and clubs including cross country, National Honor Society, Student Council, Phi Delta Y, and the Whale Tail Club.

"the best portions of a good man's life are his nameless unremembered acts of kindness and love."

He feels that, "MAΘ is a close-knit group of interested people who all serve to urge each other to new levels of mathematical understanding ... and promote a greater love for mathematics." Janet Marderness, his sponsor and teacher, sums up: "There is a 'love of learning' attitude that Alex exudes, a quality that so refreshing and comforting to teachers who rarely see that thirst for knowledge in today's students." Each local valedictorian was asked to make a statement concerfing his philosophy of life by the Fort Myers News Press. Alex's statement was true Alex: "the best portions of a good man's life are his nameless unremembered acts of kindness and love."

Each summer the MAΘ Governing Council faces a difficult decision in choosing the winners of the Huneke and Kalin awards. In 1990, they selected two exceptional people. Congratulations to Grace Mutz and Alex Hartemink!

Competition Corner

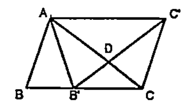
This column is devoted to information on the new Mandlebrot Competition conceived by Mu Alpha Theta alumni and the solutions to the Power Questiion of the 1990 ARML Competition given in the October1990 issue.

> If $y = x^n$ where x and n are natural numbers with $n > 1$, how many distinct values of y are less than 1991?

This problem was one of eight in the Individual Round of the first Mandlebrot Mathematics Competition. Conceived by recent Mu Alpha Theta alumni Richard Rusczyk, Sandor Lehoczky, and Samuel Vandervelde [currently undergraduates at Princeton, Princeton, and Swarthmore, respectively], the competition consists of five two-day rounds spread throughout the year. Each round has two forty- minute parts: an Individual Round with eight questions of varying difficulty in which each answer is a number or expression and a Team Round of three to five proofs related to a single problem situation. Participants are given the topic of the team round in advance. Prizes are pizza parties and money. Try to work the sample problems given here.

This trio made a special effort to create problems that are accessible to the inexperienced problem solver while giving them the opportumity to improve their proof skills and their ability to function as a part of a team. The Log applauds the initiative of these three young men and wishes them every success. For more information about the competition, contact Greater Testing Concepts, P.O.Box 471, Princeton University, Princeton, NJ 08544; phone (609) 258-7748.

Sample Team Test



In the above diagram, AB = AB', AB < AC, and 60° < angle B < 90°. To form the above diagram, we take isosceles ΔABC, AC = BC, and construct another congruent ΔAB'C' so that B' is on BC as in the diagram.

i) Show that a single circle passes through A, B', C, C'.

ii) Show that line AB is tangent to the circle at point A.

iii) Show that ΔB'DC is an isosceles triangle.

iv) Prove that ABCC' is a parallelogram.

v) Find the ratio of the area of ΔADB' to the area of quadrilateral AB'CC in terms of B'C and AC'.

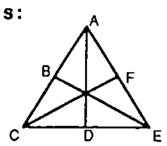
If the students are unable to prove part (i), they may assume it is true to do parts (ii)-(v).

The Mandelbrot Competition

Team Test Round 1

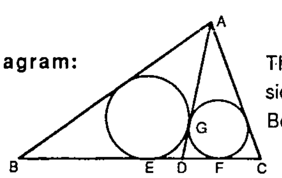
October, 1990

Facts:



Ceva's Theorem: Lines AD, BE, and CF are concurrent if and only if (AB)(CD)(EF) = (BC)(DE)(FA).

Diagram:

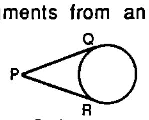


The two incircles (circles tangent to all three sides of a triangle) are tangent to BC at E and F. Both circles are tangent to AD at G.

Problems:

i. (3 points) Prove that tangent segments from an exterior point to a circle are equal:

Show PQ = PR.



ii. (3 points) Suppose that in ΔABC, AD is drawn such that the incircles of ΔABD and ΔACD are each tangent to AD at G (see diagram above). If AB=c, AC=b, and BC=a, find the length of BD in terms of a, b, and c.

iii. (4 points) Let the radii of the two circles in the diagram be r and s. Show that the length of DF is $\sqrt{rs}$.

iv. (5 points) Let line *l* be the angle bisector of angle ABC, line *m* be the angle bisector of angle ACB, and line *n* be the perpendicular to BC at point D. Prove that lines *l*, *m*, and *n* are concurrent, i.e. all three meet at a single point.

v. (5 points) Suppose that in ΔABC, points H and I are defined on segments AC and AB in the same manner as D was defined on BC in part ii. Prove that the lines AD, BH, and CI are concurrent.

The Mandelbrot Competition

Individual Test Round 1

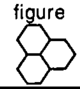
1) A basketball player scores an average of 18.6 points per game for five games. How many points must he score in the next game to raise his average to 20 points per game?

2) The point (1,1,1) is rotated 180° about the y-axis, then reflected through the y-z plane, then reflected through the x-z plane, then rotated 180° about the y-axis, then reflected through the x-z plane. What are the coordinates of the point now?

3) Find the sum of the squares of the solutions of $ax^2 + bx + c = 0$ in terms of a, b, and c.

4) Given that $y=x^n$, where x and n are integers with $x>0$ and $n>1$, how many distinct values of y are less than 1991?

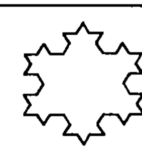
5) A configuration is made of congruent regular hexagons, where each hexagon shares a side with another hexagon. What is the largest integer k, such that the figure cannot have k vertices? For example, this figure has 13 vertices.



6) Find the volume of the region in space defined by $|x + y + z| + |x + y - z| \le 8$, where $x, y, z \ge 0$.

7) Find all ordered pairs of non-negative integers (b,c) such that $\lim_{n\to\infty} (u_n/u_{n-1}) = 3$, if $u_0 = u_1 = 1$ and $u_n = bu_{n-1} + cu_{n-2}$.

8) Trisect each side of an equilateral triangle of side length 2 and construct an equilateral triangle on the center third of each side. Omit the side of each new triangle that was once a part of the original triangle. Repeat the trisection, construction, and omission steps on each segment formed above, forming the figure on the right. What is the area of the figure formed by repeating these steps indefinitely?



SAMPLE SOLUTION TO ARML POWER QUESTION -- 1990

I. (A) Since $(a)^3+(2a)^3=9a^3$, let $a=k^2$ and $b=2k^2$. Then $(k^2)^3+(2k^2)^3=9k^6=(3k^3)^2$.

(B) Letting $c=d/r$ and $e=dr$, we get $a^3+b^3=d^3$, which has NO SOLUTIONS.

(C) $a^3+b^3=[a+b][(a+b)^2-3ab]$. Let $3ab=k^2$. Then $a^3+b^3=[(a+b)-k][a+b][(a+b)+k]$. Note: Choosing $b=3$ and $a=m^2$ for example, the smallest factor will be m^2+3-3m, which is required to be positive. Therefore $m^2+3-3m \geq 1 \Rightarrow m^2-3m+2 \geq 0$, which is easily satisfied (e.g. $m=2$ produces $a=4, b=3, k=6$).

(D) $a^3+b^3=[a+b][(a+b)^2-3ab]=3p$. Then 3 must divide one of the factors in brackets. But note that it will then also divide the other factor. Therefore 9 divides the product, so $3p$ is a multiple of 9. NO SOLUTIONS for p a prime greater than 3. [Note that for $a=2$ and $b=1$, $p=3$.]

II. (A) If there are solutions, then clearly <u>a</u> and <u>b</u> have the same parity (either both are even or both are odd). If both are even, their cubes are divisible by 8; dividing through by 8 as many times as possible, we eventually reach the equation $m^3+n^3=2^d$, where m and n are odd. Thus we need only consider the equation $a^3+b^3=2^c$ where <u>a</u> and <u>b</u> are both odd (and $a>b$). Now $a^3+b^3=(a+b)(a^2-ab+b^2)=2^c \Rightarrow$ each factor is a power of 2. Since a^2-ab+b^2 is the sum of three odd numbers, it is odd; but the only power of 2 that is odd is $2^0=1$. But $a>b \Rightarrow a^2-ab+b^2=(a-b)^2+ab>1$. Therefore there are NO SOLUTIONS.

(B) Let $a=3^m$ and $b=2\cdot 3^m$. Then $(3^m)^3+(2\cdot 3^m)^3=9\cdot 3^{3m}=3^{3m+2}=3^c$.

(C) If p divides a or b, it will divide the other also. This leads to the equation $k^3p^3+m^3p^3=p^c \Rightarrow k^3+m^3=p^d$. This reasoning can be repeated until neither term on the left is divisible by p. Therefore we need only consider $a^3+b^3=p^c$, where p does not divide <u>a</u> or <u>b</u>.
$a^3+b^3=[a+b][(a+b)^2-3ab]=p^c$. Noting that the second factor in brackets is greater than 1 [since it is equal to $(a-b)^2+ab$], we see that each bracketed factor must be a power of p greater than 1. But $p|a+b$ and $p|(a+b)^2-3ab \Rightarrow p|3ab \Rightarrow p|ab \Rightarrow p$ divides either a or b. Contradiction. NO SOLUTIONS. [Note that allowing $p=3$ destroys the validity of the ending. Also note that this proof is basically equivalent to the proof of IIA, but is more general. Finally, note that (even allowing $a=b$) the sum of two cubes can never equal a prime greater than 2.]

III. (A) Let p be the largest prime less than c. $p|a$ and $p|c! \Rightarrow p|b^3 \Rightarrow p|b \Rightarrow p^3|a^3$ and $p^3|b^3$. Then $p^3|c! \Rightarrow c \geq 2p$ (actually, $c \geq 3p$). But by Bertrand's Postulate (now a theorem), there is always a prime between n and 2n (for $n \geq 2$). Therefore there is a prime between p and 2p, and it is less than c. Contradiction. Therefore $p \nmid a$.

(B) In this solution, we will sacrifice a uniform approach to illustrate several different methods. We will show that a+b is divisible by 2,3,5,11, thus establishing divisibility by 330; note that these primes divide c!, so they divide a^3+b^3 .

(1) Since a^3+b^3 is even, <u>a</u> and <u>b</u> must have the same parity. Then a+b is even, so $2|a+b$.

(2) $a^3+b^3=[a+b][(a+b)^2-3ab]$. Either 3 divides the first factor, or it divides the second factor [in which case it must divide $(a+b)^2$ and therefore a+b]. Thus $3|a+b$.

(3) If $5|a$, it must divide b, so $5|a+b$. Suppose 5 does not divide <u>a</u> or <u>b</u>. Fermat's Little Theorem says that if p is a prime and $(a,p)=1$, then $a^{p-1} \equiv 1 \pmod{p}$. Thus, for example, $a^4 \equiv b^4 \equiv 1 \pmod{5}$. Now $a^3+b^3 \equiv 0 \pmod{5} \Rightarrow$ [multiplying by ab] $a^4b+ab^4 \equiv 0 \Rightarrow b+a \equiv 0$, so $5|a+b$. This approach can be applied for each prime under consideration here, although (mod 11) requires additional clever manipulation.

(4) Consider a number of the form 11k+x, and let its cube be of the form 11K+y, $0 \leq x,y < 11$. A table comparing x and y is as follows:

x	0	1	2	3	4	5	6	7	8	9	10
y	0	1	8	5	9	4	7	2	6	3	10

. Note how when two y values add up to 11 (or 0), the corresponding x values add up to 11 (or 0). Thus since $11|a^3+b^3$, $11|a+b$. This approach can be applied for each prime under consideration. [It also works for products of some of these primes, such as 6 and 10. It will work to show that $17|a+b$, if c is large enough. Try it for divisibility by 7 to see how it fails.]

The question of whether $a^3+b^3=c!$ has solutions for $c>2$ seems to be an open question.

Solution: Since there are many ways to throw at least one six in 4 rolls of a die, it is easier to consider the **complementary event** of throwing no sixes. This probability is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296}$ and so the probability of throwing at least one six is $1 - \frac{625}{1296}$ or $\frac{671}{1296}$. Thus $X = \frac{671}{1296}(\$100) + \frac{625}{1296}(-\$100)$. He can expect to win about $3.55 in each game.

Problem 1 Compute the expected value in the second game.

Problem 2 An early form of an eastern lottery was a scratch-off game with the following prizes:
1 - $5000 prize; 18 - $200 prizes; 120 - $25 prizes; and 270- $20 prizes. It costs $ .50 to play the game. Compute your expected loss each time you played this game.

Problem 3 In another PICK 3 game, you select a 3-digit number from 000 - 999. On a straight ticket, you win $500 if you match the winning number. On a box ticket, you win $80 if your number has the same digits as the winning number. It costs $1 to play each game. Compute your expected loss each time you play each of these games.

Problem 4 In the CARDS game, you select one of the thirteen cards in each of the four suits. If you match all 4 cards, you win $1000; match 3 wins $100, and match 2 wins $10. It costs $1 to play the game and you can win only one prize per ticket. What is your expected loss on this game?

Problem 5 In the KICKER game, you are given a computer-chosen 6-digit number from 000000 - 999999. You win $10 if you match the first 2 digits of the winning number; $100 for the first 3 digits; $1000 for the first 4 digits; $5000 for the first 5 digits; and $100,000 for all 6 digits. It costs $1 to play the game and you can win only one prize per ticket. What is your expected loss on this game?

Problem 6 The odds of winning the MEGABUCKS game are given in the table. It is a scratch-off game. Can you determine the number of tickets sold in each game? If not, what additional data do you need to know? Assume a reasonable value for each piece of data and compute your expected loss playing the game.

You are also encouraged to compute your expected loss in any of the games given in the two articles cited above. My home state of Texas is considering a state lottery. Design an interesting lottery game that returns 50% of the money wagered as prizes. Send it to me, The most interesting games will be published in the Log and possibly sent to the governor of Texas.

The Alphabet Lottery

Iowa recently added a bonus to its scratch-off lottery ticket. A perforated attachment to the regular ticket adds a second game listing five letters hidden under a scratch-off surface. Winners are determined by 16 separate television drawings on which five different lettered balls are randomly selected in a specific order. If Nancy purchased a lottery ticket with this added feature, she will win $25,000 if her five letters match, in order, the five letters in the television drawing. She has 16 chances to win and it costs 25¢ for the bonus ticket. What is Nancy's expected loss in this bonus game? [Hint: Combine the two ideas in Great Expectations.]

Contributed by Bonnie H. Litwiller and David R. Duncan, Professors of Mathematics at the University of Northern Iowa in Cedar Falls, Iowa. [Ans. about 20¢]

Odds of Winning

Overall odds of winning a prize are 1 in 3.96

Entry	1:300.30
Free Ticket	1:7.54
$2	1:10.27
$5	1:65.22
$25	1:375.94
$50	1:1,428.57
$100	1:3,030.30
$1,000	1:20,000
$10,000	3 per game
$15,000	2 per game
$100,000	1 per game
$2,000,000	1 per game