Attend the 1991 Convention in Huntsville, Alabama p. 4

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A Tale of Two Cubics

The pursuit of the roots of the cubic polynomials $p(x) = x^3 - 3x + 1$ and $q(x) = x^3 - 3x - 1$ leads us along two disparate paths concerning: (1) the impossibility of constructing a regular nonagon, and (2) the Fixed Point Iteration algorithm for approximating real roots. The two cubics are closely related since their graphs are half-turn images of each other.

FIXED POINT ITERATION

According to its graph, the cubic polynomial $p(x) = x^3 - 3x + 1$ has three real roots: $0 < r_1 < 1$; $1 < r_2 < 2$; $-2 < r_3 < -1$. Although we could find them using the ZOOM feature of a graphing calculator or a root finding program, let us investigate an algorithm that can be performed on a basic scientific calculator. To find a value of x satisfying $x^3 - 3x + 1 = 0$ is equivalent to finding an x satisfying $x = g(x) = \frac{x^3 + 1}{3}$. To use this choice of g(x) to approximate r_1, first choose an initial guess, say, $x_0 = .5$. Then repeatedly use the recursion equation $x_{n+1} = \frac{x_n^3 + 1}{3}$ until $x_{n+1} = x_n$ [on the calculator] or the sequence $\{x_n\}$ diverges. Table 1 gives these sequences for various values of x_0.

$x_0 = .5$	$x_0 = 0$	$x_0 = 1$	$x_0 = 1.5$
.375	.333333	.666666	1.45833
.35091	.345689	.432000	1.36716
.347737	.347102	.360225	1.18514
.347350	.347273	.348915	.888196
.347303	.347296	.347492	.566897
.347297	.347296	.347320	.349062
.347296		.347299	.348087
.347296		.347297	.347392
		.347296	.347308
		.347296	.347298
			.347297
			.347296
			.347296

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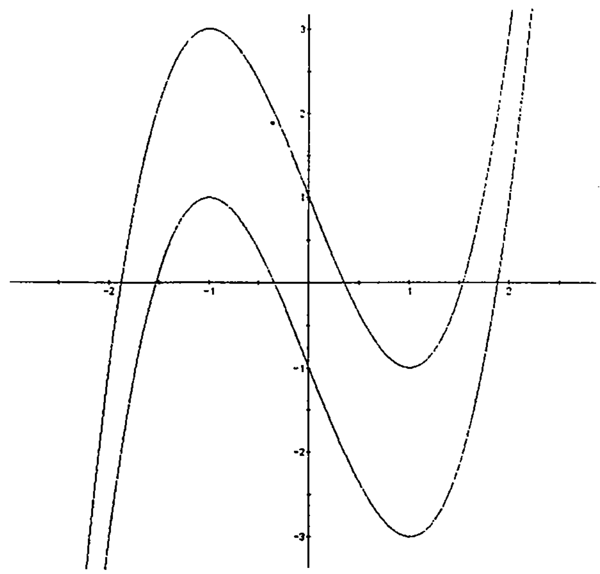

REGULAR HEPTAGONS and NONAGONS

Ali R. Amir-Moéz
Texas Tech University, 79409

Karl Friedrich Gauss (1777-1855) proved that dividing a circle into n congruent arcs, where n is a prime number, has ruler and compass construction if

$$n = 2^{(2^K)} + 1.$$

In particular, at the age of eighteen he gave a ruler-compass construction of dividing a circle into seventeen equal arcs.

Sharof-al-Din al-Tusi a student of Abu Raihan Al-Biruni (973-1048) studied the cubic equation $x^3 - 3x = 1$ in connection with al-Biruni's division of a circle into nine equal arcs.

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"When Two Things Are Created Equal" - Rule XIX for Solving Problems

According to Polya[1], "Rene Descartes was one of the very great." In his Rules for the Direction of ther Mind, Descartes planned to present a universal method for the solution of problems. Although there are gaps and impracticalities in his approach, there is much valuable advice as well. One worthwhile suggestion is [a paraphrase of] Rule XIX: [To set up an equation to solve a problem], **express the same quantity in two different ways.**

We examine several examples involving this rule. You are encouraged to complete each one and the problems that follow.

Example 1 Find the equation of the line containing the points (2,3) and (4,7).

Solution Let (x,y) be an arbitrary point on the line. Express the slope of the line in two different ways: $\frac{7-3}{4-2} = \frac{y-3}{x-2}$. Check the answer using (4,7) in the right side.

Example 2 Airline schedules allow an additional 30 minutes flight time to fly from Chicago to Phoenix than from Phoenix to Chicago. Assuming the average airspeed of a plane is 525 mph and the distance from Chicago to Phoenix is 1453 miles, find the average rate of the "jet stream" - a wind that blows from west to east.

Solution Let the rate of the wind = x. Express the difference of the flight times in two different ways:

$$\frac{1453}{525-x} - \frac{1453}{525+x} = \frac{1}{2}$$

Example 3 A dealer has two kinds of nuts: one costs 90 cents per pound and the other 60 cents per pound. He wishes to make 50 pounds of a mixture that will sell for 72 cents per pound. How many pounds of each kind should he use?

Solution Let the number of pounds of the two kinds of nuts = x and y. Express the total weight and the total price of the mixture in two different ways:

$x + y = 50$ and $90x + 60y = 72 \cdot 50 = 3600$.

Continued on page 4

CONTEST: REPORT and ALGEBOGGLE

Congratulations to the triumvirate of **Pramod Achar, Andrew Chang,** and **Sameer Shah** of White Station High School in Memphis, Tennessee who achieved the highest score of 884 points in the GeoBoggle Contest. Their strategy was to include several forms of each of 15 key words [e.g. octagon, octagonal, octagons, octagonally]. The second highest score was 279 points by **Francois Devred** and **Josh Freeman** of Lakota High School in West Chester, Ohio who used 12 key words in their entry. **Jason Moore** of South High School in Kingsport, Tennessee captured 3rd place with 205 points. Sincere thanks to the 20 other students from around the country who submitted entries. [Some words that could not be found in the American Heritage Dictionary were disallowed.]

In this contest [see Oct. 1990 issue], design a 5-by-5 BOGGLE ® board that contains a set of **algebra** words with the highest score. You are to designate one letter for each of the 25 squares. The same letter may appear in more than one square. Words are formed from adjoining letters - either horizontally, vertically, or diagonally. Letters must be joined in sequence to spell a word that could be found in an algebra book. Proper nouns, abbreviations, contractions, and hyphenated words are not permitted. **Only one form of a word may be used.** No letter [from a given square] may be used more than once in a single word. Each word must have at least four letters. Scoring will be as follows:

No. of letters	4	5	6	7	8	9	10 or more
No. of points	1	2	3	5	8	13	21

Your entry should consist of:
1. Your 5-by-5 board with 25 letters.
2. An alphabetical list of the words that can be formed using your board.
3. The total score of the words formed in step (2).

Send your entry to the Editor [see page 2 for address]. Entries must be received by November 15, 1991. The entry with the highest score will be the winner.

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The Mathematical Log is the official publication of Mu AlphaTheta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association ofAmerica (MAA)and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to the Editor: Thomas R. Butts,University of Texas at Dallas,P.O. Box 830688 Richardson,TX 75083-0688, or to Mu AlphaTheta National Office, 601 Elm Ave., Room 423, Norman, OK 73019.

1991 √At the Root of It All

Deborah Patonai Phillips, Activities Editor

Two MAΘ Newcomers

The Mu Alpha Theta family is constantly changing and growing. The faces of the students vary from year to year, but a majority of the dedicated sponsors remain, adding stability and experience to their chapters. √At the Root of It All is proud to present two new members of this family. One is a sponsor from a little island off the coast of Maine; the other is the new editor of the of The Math Log.

First, let's meet the new editor, Dr. Thomas Butts from Dallas, Texas. A relative newcomer to MAΘ , his previous experience is limited to speaking at several initiation ceremonies. Thanks to some friendly persuasion from former MAΘ president Pam Drummond, Tom became the new editor of The Log, - "a classic case of baptism under fire".

Involved in mathematics education for "n years where n > 20", he received his M.S. and PhD in mathematics from Michigan State University. Spending most of his career in teacher preparation and inservice programs, he was also a certified secondary teacher in Illinois and recently taught a small class of 8th graders for almost two years. Currently completing his tenth year at the University of Texas at Dallas, he previously taught at Case Western Reserve University, Ohio State Unversity and his alma mater.

As a teacher, one of his trademarks is beginning each class with a "Question of the Day". Some examples:

1. What do the following have in common?
   a. Twins, Rangers, Angels
   b. Christmas, Easter, Melville
   c. John Adams, Thomas Jefferson, James Madison
2. What is the longest running network televison show?
3. Who won the most Oscars?

[Answers, in case you're stumped, appear at the end of this column.]

His favorite courses to teach are entitled "Usual and Unusual Problems Using ____ " where the current choices for the blank include Algebra, Geometry, Precalculus,Probability/Statistics, and Computers and Calculators. Besides collaborating on a textbook in intermediate algebra, and on a series of textbooks for grades 6,7, and 8, he has written Problem Solving in Mathematics [a long time ago], and two monographs on problem solving for the Texas Education Agency. He is currently embarking on a series of secondary textbooks on integrated mathematics.

Another of his long time loves [in addition to his family and bridge] is mathematics competitions. Since 1980, he has served as a panelist or committee member for the American Mathematics Competitions. From 1984 - 1990 he headed the subcommittee responsible the American Junior High School Mathematics Examination.

As a "novice" editor, Tom would welcome your advice and suggestions on ways to improve The Log. Send him your ideas, recommendations, stories, and articles!

Our other newcomer is Lee (Pete) Pedersen from Vinalhaven Island off the Maine coast. Even though Vinalhaven has only 50 students, it is becoming well known in mathematics due mainly to Pete's efforts. In 1987 the team started to compete at the state level - placing last. Within three years they have become a trophy-winning team. Pete said he would work "night and day" to help the team improve. Students took him at his word. Some arrive at his home at 4:30 a.m. and work until the start of school. Others come in the early eveming and work late into the night.

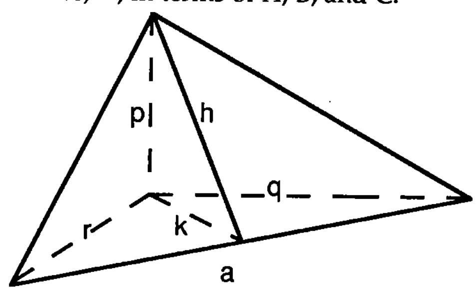
Even though Vinalhaven has only 50 students, it is becoming well known in mathematics due mainly to Pete's efforts.

Chartered in 1989, the chapter has already become a vital force within the area. They are campaigning throughout central Maine with the goal of seeing at least nine more MAΘ chapters formed this year. Even though the club has only 8 members, they are quite active. Last year they sponsored a math meet for the Maine Island Schools. The members love to travel [to competitions and meetings] and to eat - always having a "wicked keen" time. They have attended the last two conventions in Tampa and DeKalb. Arriving early in Chicago last summer, they celebrated Pete's birthday in style by touring the city in a limo and enjoying a great meal.

They have volunteered to host the 1995 national convention. To be held in Portland, Maine, this enthusiastic group has already planned a trip to L.L. Bean, a lobster bake at an offshore island, and a cruise on the bay. Make your plans early to attend!
Even though Tom Butts and "Pete" Pedersen are new faces , they both seem ready and eager to commit their talents and energy to our family. Welcome aboard!

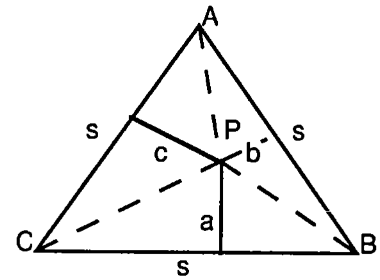
[Answers to the Question of the Day: 1. major league baseball teams named after states (not cities), islands, presidents who died on July 4; 2. Meet the Press; 3. Walt Disney]

Example 4 In a tetrahedron that possesses a trirectangular vertex , the areas of the three faces containing this vertex are A, B, and C. Find the area of the fourth face, D, in terms of A, B, and C.



Solution $D=\frac{ah}{2}$ and $h^2 = k^2 + p^2$. Express the area of the bottom face in two ways to find k: $A = \frac{1}{2}ak$. then $D^2 = a^2h^2$. Continue using $a^2=r^2+q^2$, $B=\frac{1}{2}rp$ and $C=\frac{1}{2}pq$. This is a 3-dimensional analog of the Pythagorean Theorem.

Example 5 Viviani's Theorem [1636] : For a point P inside an equilateral triangle ΔABC, the sum of the distances from P to the sides of ΔABC = the altitude h.



Solution Express the area of ΔABC in two ways:
$\frac{s^2}{4}\sqrt{3} = \frac{1}{2}as+\frac{1}{2}bs+\frac{1}{2}cs$ or $a+b+c=\frac{\sqrt{3}}{2}s$.

Try to apply Descartes Rule when you solve each of these problems.

Problem 1 Find the length of the altitude to the hypotenuse of a right triangle with legs 9 and 16.

Problem 2 In a triangle, find the length of the bisector of the 120° angle with included sides of lengths 6 and 8. [Hint: Use trigonometry]

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Blast Off to the 21st Mu Alpha Theta National Convention

The 1991 Mu Alpha Theta National Convention will be held Thursday, August 1 through Tuesday, August 6 at the Huntsville Hilton Hotel in Huntsville, Alabama. The highlight of the convention will be a Space Camp Action Tour at the Huntsville Space & Rocket Center for each participant at the convention. The better part of two days will be spent at the Space Camp.

The 1991 Convention will have many exciting speakers including a local astronaut and the national MAΘ President, Dr. John Kenelly. There will be several rounds of mathematics competitions including an interschool test, a written and ciphering competition for all students, ciphering rounds for school teams and state teams and topic tests in 16 different areas. A visit to the campus of the University of Alabama in Huntsville is also planned. Selected members of the UAH faculty are planning a special program for all convention participants.

A concert on Sunday night and a very exciting awards banquet on Monday night is planned. There will be an atmosphere of fun and excitement at the convention so that students and sponsors will not only participate in interesting mathematics activities, but will also enjoy this trip as if it were a vacation.

Divisions: Individual, school and state competitions will be divided into two divisions, Theta and Alpha. The Theta Division is open to any student who has only completed Algebra 1, Geometry, Algebra 2 or Algebra 2/Trigonometry. The Alpha Division is open to all students.
Interschool test: This is a 30-question, 90 minute free response test which will be worked on by all students from a particular school. At the end of the time limit, each school is to submit one set of answers to the proctor.
Written Test: A 60-minute, 25 question multiple choice test will be given in each division. All students are expected to compete in one of the two divisions. Scoring will be on the basis of 4 points for each correct answer minus 1 point for each incorrect answer. A blank response neither receives nor loses any points.

Ciphering Test: A 12-question ciphering round will be held in each division and all students are to participate in one of the divisions. 8 points will be awarded to studetns who answer the question in minute one, 5 points to those who answer in minute two and 2 points in minute three.
School Team Ciphering: The top four students in each division from each school (based on total Round 2 Written and Round 3 Ciphering points) will compete in a 12-question team round. Scoring is the same as the ciphering test above.
State Team Ciphering: This is the same as the school team competition, but it will feature the top four students from each state based on Round 2 Written and Round 3 Ciphering points.
Topic Tests: Sixteen topic tests will be administered in four sessions. These tests will consist of 25 multiple choice questions and a time limit of 40 minutes will be imposed. The scoring will be the same as the written test.

The registration fee is $275 per person, if paid by April 30, $290 per person if paid after April 30. The package includes fees for all meals, hotel accommodations, space camp, transportation to and from the airport, entertainment, etc. Contact American Airlines for Meeting Saver Fare Discounts. To receive more information about the
Huntsville Space & Rocket Center MAΘ Convention
mail the form below to: Randy Long
Grissom High School
7901 Bailey Cove Road
Huntsville, AL 35802

Please send me more information about the
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In this note we study Gauss's techniques obtaining a third degree equation related to dividing a circle into seven equal arcs and we also employ trigonometric identities to obtain al-Biruni's equations related to the construction of a regular nonagon.

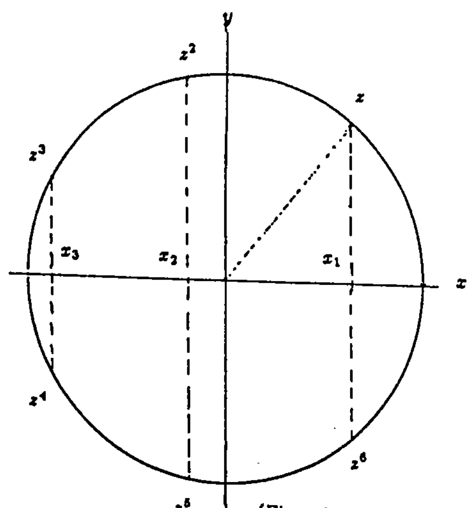
1. Regular Heptagon: We choose a unit circle, i.e., a circle of radius one (Fig.1). Dividing this circle into seven equal arcs is related to the equation

$$z^7 - 1 = 0. \quad (1)$$

We can factor (1) as

$$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0. \quad (2)$$

Note that the root $z=1$ corresponds to (1,0) in (Fig. 1). Other roots are of the form



(Fig. 1)

$$z_k = \cos\frac{2\pi+2k\pi}{7} + i\sin\frac{2\pi+2k\pi}{7}, \quad k = 0,1,\cdots,6. \quad (3)$$

Let the roots be $z_0 = 1$, and $z_1 = z, z_2, \cdots, z_6$. then

$$z_k = z^k, \quad k = 1,\cdots,6. \quad (4)$$

Let z^* denote the conjugate of z. Then

$$z_h^* = z_{7-h}, \quad h = 4,5,6. \quad (5)$$

Now if we let

$$z_k = x_k + iy_k, \quad (6)$$

then

$$z_k + z_{7-k}^* = 2x_k, k = 1,2,3\,. \quad (7)$$

2. The Cubic Equation Related to the Heptagon: From (7) we observe that

$$\begin{cases} z_1+z_6 = z+z^6 = 2x_1, \\ z_2+z_5 = z^2+z^5 = 2x_2, \\ z_3+z_4 = z^3+z^4 = 2x_3. \end{cases} \quad (8)$$

Therefore, we get

$$2(x_1+x_2+x_3) = z+z^2+z^3+z^4+z^5+z^6 = -1,$$

$$\begin{cases} 4x_1x_2 = (z+z^6)(z^2+z^5) = z^3+z^6+z+z^4 = 2(x_1+x_3), \\ 4x_1x_3 = (z+z^6)(z^3+z^4) = z^4+z^5+z^2+z^3 = 2(x_2+x_3), \\ 4x_2x_3 = (z^2+z^5)(z^3+z^4) = z^5+z^6+z+z^2 = 2(x_1+x_2). \end{cases} \quad (9)$$

So

$$x_1x_2+x_1x_3+x_2x_3 = x_1+x_2+x_3 = -\frac{1}{2}. \quad (10)$$

Finally

$$8x_1x_2x_3 = (z+z^6)(z^2+z^5)(z^3+z^4) =$$

$$1 + \ (1+z+z^2+z^3+z^4+z^5+z^6) = 1,$$

which implies

$$x_1x_2x_3 = \frac{1}{8}. \quad (11)$$

From (9), (10), and (11) we conclude that x_1, x_2, x_3 satisfy the cubic equation

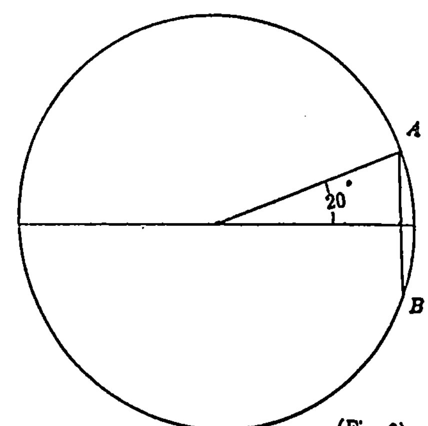
$$x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8} = 0. \quad (12)$$

Since (12) can not be factored, there is no ruler-compass construction for the regular heptagon.

3. Regular Nonagons: If we divide a circle into nine equal arcs, the central angle corresponding to each arc is 40° (Fig. 2).

Al-Biruni first chose AB=$2\sin 20°$, for the unknown. Observing that

$$\sin 60° = 3\sin 20° - 4\sin^3 20°, \quad (13)$$



(Fig. 2)

and setting AB=x, from (13) he obtained

$$\sqrt{3} = 3x - x^3. \quad (14)$$

Having $\sqrt{3}$ in (14) did not please al-Biruni. So he chose $2\cos 20° = x$ and employing the identity

$$\cos 60° = 4\cos^3 20° - 3\cos 20°, \quad (15)$$

he obtained

$$x^3 - 3x - 1 = 0. \quad (16)$$

Since (16) can not be factored, there is no ruler-compass construction for the regular nonagon.

$x_0 = 1.55$	$x_0 = -.9$	$x_0 = -1.85$	$x_0 = -1.9$
1.57463	.090333	−1.77720	−1.953
1.63473	.333579	−1.53775	−2.14972
1.78952	.345706	−.878766	−2.97815
2.24359	.347105	.107130	−8.47144
4.09783	.347273	.333743	−202.3183
23.27051	.347294	.345725	diverges
4200.7873	.347296	.347108	
diverges	.347296	.347274	
		.347294	
		.347296	
		.347296	

Table 1

Thus for [approximately] $-1.85 < x_0 < 1.5$, the sequence converges to the root, r_1. These results raise (at least) two questions:

(1) When does this algorithm work?
(2) How can we find the other two roots?

To explain this behavior, let us see what happens when we apply $g(x) = \frac{x^3+1}{3}$ to small intervals about two successive guesses. Take $x_0 = .5$. Now g([.45,.55]) = [.3637,3888] - an interval of length .025 or one fourth the length of the original interval [.45,.55]. Now $x_1 = .375$ and g([.325,.425]) = [.3448,.3589], a still smaller interval.

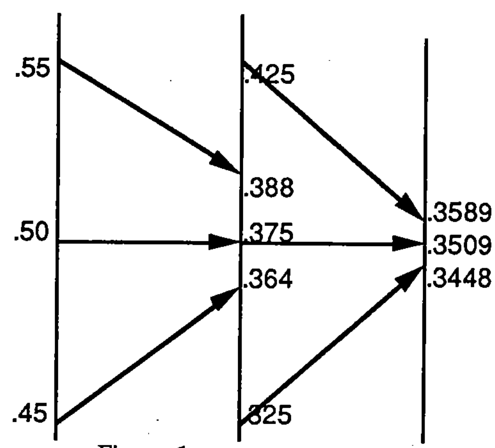


Figure 1

If we perform a similar set of computations with the initial guess $x_0 = 2$, then we obtain Figure 2.

It seems reasonable, then, that convergence occurs when these intervals get smaller. If x_n is a guess for a root, then this occurs if $|g(x_n + h) - g(x_n - h)| < 2h$ or

$$\left| \frac{g(x_n + h) - g(x_n - h)}{2h} \right| < 1.$$ In most cases,

$$\lim_{h \to 0} \frac{g(x_n + h) - g(x_n - h)}{2h} = g'(x_n),$$ so the following theorem is plausible.

Theorem: The Fixed Point Iteration algorithm converges if the absolute value of the first derivative $|g'(x)| < 1$ for values of x near the initial guess x_0.

If $g(x) = \frac{x^3+1}{3}$, then $g'(x) = x^2$. Convergence is guaranteed for $-1 < x_0 < 1$. Experimentally we saw that convergence occurred for the larger interval (−1.85, 1.5).

Exercise 1 Use the algorithm to obtain the two remaining roots using other choices for g(x), e.g. $g(x) = \sqrt[3]{3x - 1}$, $g(x) = \frac{3}{x} - \frac{1}{x^2}$.

Exercise 2 Find the base of an isosceles triangle whose area is 2 and whose perimeter is 8.

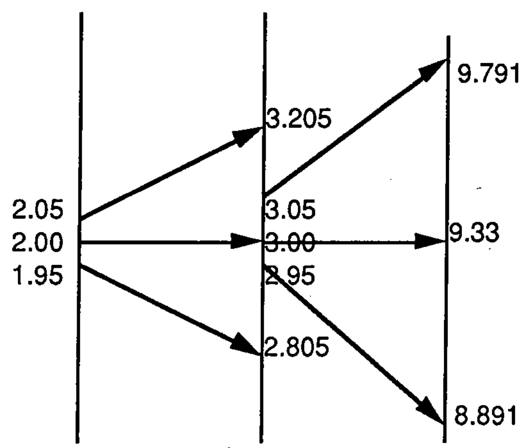
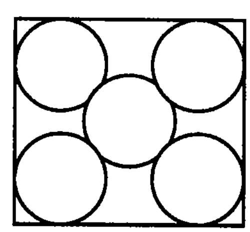


Figure 2

Problem 3 Using five right triangles with legs of lengths 1 and 2, form a right triangle that is similar to each of the five smaller triangles.

Problem 4 Inside a square of side a are five congruent non-overlapping circles of radius r. One circle is concentric with the square and touches the other four circles, each of which touches two sides of the square. Express r in terms of a.



Problem 5 Find the hypotenuse of a right triangle in terms of its area A and its perimeter P. [Hint: If a and b are the legs, express $(a+b)^2$ in two ways.]

1. Mathematical Discovery, George Polya, 1962