Thanks to Don Allen ....Carol's Cubicle p. 2

The Mathematical Log

Volume 34 Number 3, October 1990

BUILDING BOXES, OR WHEN HAVE YOU SOLVED A PROBLEM ?

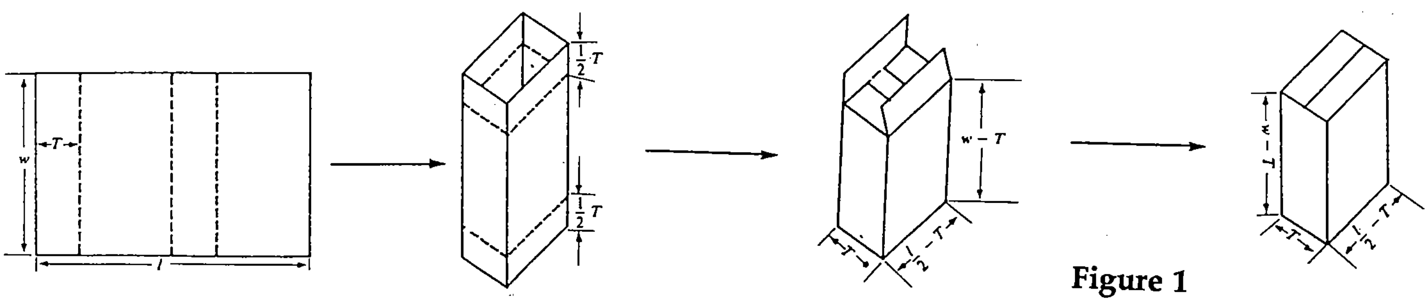


Figure 1

You can make a "popcorn box" from a rectangular piece of heavy paper by followng the sequence of steps in Figure 1. Consider the following

PROBLEM Suppose l = 24 units and w = 12 units.

a. Find the dimensions of all boxes whose volume is 200 cubic units

b. Find the dimensions of the box of maximum volume.

What constitutes an "acceptable" solution to this problem, or, for that matter, to any problem? Let us examine several solutions to this problem. They are in the spirit of the levels of differentiation of the core curriculum suggested in the new [1990] Curriculum Standards of the National Council of Teachers of Mathematics. [c.f. pp. 129 - 135]

SOLUTION 1
Let T = x in the figure. Express the volume, V, in terms of the height, x, by

$V = lwh = (12 - x)(12 - x)x = x(12 - x)^2$.

Use a calculator to make a table of values over a sensible range of values for x, the integers $0 \le x \le 12$. [Figure 2]. By examining the table, V = 200 when x = 2 and when $x \approx 6.5$. The maximum value of V is 256 for a box with dimensions 8x8x4.

SOLUTION 2
Write a computer program to produce more accurate tables of values, [Figure 3]. By examining the table, V = 200 when x = 2 and when $x \approx 6.42$. The maximum value of V is 256 for a box with dimensions 8x8x4.

Exercise Expand this program to one that prints the maximum value of V and the value of x at which it occurs accurate to 2 decimal places.

HEIGHT	VOLUME
0	0
1	121
2	200
3	243
4	256
5	245
6	216
7	175
8	128
9	81
10	40
11	11
12	0

Figure 2

CONTINUED ON PAGE 5

PLAN NOW ....1991 CONVENTION IN HUNTSVILLE, ALABAMA

CONTEST: GEOBOGGLE

The challenge in this contest is to design a 5-by-5 BOGGLE® board that contains a set of geometry words with the highest score.

You are to designate one letter for each of the 25 squares. The same letter may appear in more than one square. Words are formed from adjoining letters - either horizontally, vertically, or diagonally. Letters must be joined in sequence to spell a word that could be found in a geometry book. Proper nouns, abbreviations, contractions, and hyphenated words are not permitted. No letter [from a given square] may be used more than once in a single word. Each word must have at least four letters. Scoring will be as follows:

No. of letters in word	4	5	6	7	8	9	10 or more
No. of points	1	2	3	5	8	13	21

Here is a sample 3-by-3 board [in which 3-letter words were allowed]:

K	T	P
I	L	E
G	N	A

This board contains the geometry words below [and possibly others]:

angle, kite, line, plane, net, ten

Your entry should consist of:

1. Your 5-by-5 board with 25 letters.
2. An alphabetical list of the words that can be formed using your board.
3. The total score of the words formed in step (2).

Send your entry to the Editor [see page 2 for address]. Entries must be received by December 15, 1990. The entry with the highest score will be the winner.

Carol's Cubicle

It gives me great pleasure that my first official act as president-elect of Mu Alpha Theta is to thank Don Allen for his ten years of service to the organization as editor of the Mathematical Log. Dr. Allen has been an inspiration to many students by introducing them to the joys of recreational mathematics. He was a convention speaker during most of his term and always had time for students and/or sponsors who wanted to discuss mathematics. Students were Don's primary interest at Mu Alpha Theta functions where he went out of his way to meet them and to listen to their ideas. So thank you Don for your dedication and service. You will be missed.

Sincerely,

Carol McGill

The Mathematical Log

Volume 34 Number 3, October 1990

The Mathematical Log is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM). Correspondence may be directed to the Editor or to Mu AlphaTheta National Office, 601 Elm Ave., Room 423, Norman, OK 73019.

1990 √At the Root of It All

Deborah Patonai Phillips, Activities Editor

Salute to the Andree Award Winners

• "My excitement for mathematics and my desire to help others are the reasons for my wanting to become a mathematics teacher. I will be using my mathematical abilities to make a difference in many people's lives."

• "The Lord blesses each newborn child by giving him his own individual and unique gifts. Some are given perfect health and others classic beauty. Still others possess great physical endurance, analytical thinking, or the calling to preach the Lord's word. I believe that God puts each person on Earth with the ability to fulfill a personal destiny. Mine is to become a mathematics teacher."

With these words, this year's Andree Award winner and runner-up summarized their reasons for wishing to become mathematics teachers. Kara Finley, a '90 graduate of Alan C. Pope High School in Marietta, Georgia, was this year's winner. Runner-up was Cindy Chao, a '90 graduate of Newman Smith High School in Carrollton, Texas. The awards are in honor of the founders of MAΘ - Richard and Josephine Andree. There is a $1000 cash prize for the winner and $300 for the runner-up. Both awards include a plaque and complimentary registration at next year's national convention.

Both excelled in the classroom.

Both young women excelled in the classroom. Kara ranked first in her class of 435 with a 4.1 GPA including several AP courses while Cindy also earned A's in her AP courses and ranked in the top 2% of her class of 540 students. According to her Algebra II teacher Murray Siegel, "Kara wrote a beautiful proof [to an extra credit problem] using a number of concepts that she melded in a elegant fashion to demonstrate the theorem's validity."

Cindy has been active in MAΘ, Key Club, SADD, marching and symphonic band, Student Council, Latin Club, National Honor Society, H.O.S.A., and President's Club. She also donated countless hours at a nursing home, a blood center, and several other community projects.

An excellent tennis player, Kara won the state championship as a sophomore and was a finalist in her junior year. She also participated in volleyball, basketball, and powder Puff football. Sometimes at the cost of missing athletic events, Kara participated in many math tournaments often being the top scorer.

She, too, found time for many extra-curricular activities including Key Club, Video Yearbook staff, Beta Club, the Fellowship for Christian Athletes, the Special Olympics, and the Hurricane Hugo relief drive.

Both possess tremendous teaching qualifications

"She [Cindy] seems destined to become a teacher," says Jean Kibler, her calculus teacher/MAΘ sponsor. Kara's sponsor Charlyn Shepherd relates: "Kara will put her heart into it [teaching] and encourage those around her to excel. Her background will bernefit and inspire her students." Both Kara and Cindy possess tremendous qualifications for becoming successful and dedicated mathematics teachers. Mu Alpha Theta wishes them well in their college endeavors and future careers.

PP's POINTS Students from South Pike School in Magnolia, Mississippi, enjoyed a year of championships. For the 1989-90 school year, they won FOUR state championships: Football, Girls' Basketball, Boys' Basketball, and MAΘ State Champs. Congratulations!

Mathematics and Music: Some Intersections

• "How do you make a duo-chord?"

• "Why does an orchestra never play in tune?"

These are only two of the questions addressed in the new Mu Alpha Theta publication: Mathematics and Music: Some Intersections by Joan Reinhalter. It is an informative account of the fascinating interplay between history, music, and mathematics - from ratios to transformations, matrices, and groups. To obtain your copy, send $2.50 plus 50¢ postage to the Mu Alpha Theta National Office. [see address on page 2]

In the News

Using a new program and time on over 1000 computers around the world, mathematicians have recently [June 1990] factored a 155-digit number as the product of exactly three primes. The number, whose factors have 7, 49, and 99 digits repsectively, was almost 50 digits longer than any such number previously factored - a remarkable increase in complexity . Far from being of only recreational interest, this factorization shows the vulnerability of many security codes. Banks and other organizations use the prime factors of large numbers as "keys" in the decoding of secret data transferred electronically.

Simple coding schemes

Cryptographs are coded messages. In a code, a numeric value is assigned to each alphabetic character, numeral, punctuation symbol, special character, and blank space. These values are often those of the ASCII code [see below]. Computers can convert characters to ASCII code and vice-versa. A coding scheme requires an encoding function, E(x), and a decoding function, D(y).

Portion of ASCII Table

A	B	C	D	E	...	X	Y	Z	space	0	...	9
65	66	67	68	69		88	89	90	32	48		57

Example Code 'MATH IS FUN' with $E(x) = 2x + 3$.

ASCII code: 77 65 84 72 32 73 83 32 70 85 78

Encoded message: 157 133 171 147 67 149 169 67 143 173 159

To decode the message, use $D(y) = \frac{y-3}{2}$. [Why?]

The decoded message is 77 65 84 72 32 73 83 32 70 85 78 as expected.

Public Key Cryptography

More sophisticated codes use encoding functions of the form $E(x) \equiv x^s \bmod pq$ where p and q are 'large' primes. [E(x) is the remainder when x^s is divided by pq .] If s = 3, for example, the decoding function is

$$D(y) \equiv y^{\frac{2(p-1)(q-1)+1}{3}} \bmod pq.$$

In this scheme, the product r = pq and s can be public. Anyone knowing the values of r and s can send a message, but only those knowing the prime factors p and q can decode it. Thus if r = pq is assumed to be "unfactorable", those using the code have confidence that their messages will not be decoded by an unauthorized person. Up to now, p and q have had approximately 50 digits. The factorization of this 155-digit number will probably cause banks and government agencies to use larger numbers for r and s even though the computations involved will be more complex and expensive.

References

1. New York Times, July 20, 1990
2. "Get the Message? Cryptography, Mathematics, and Computers", J. Reagan, Mathematics Teacher, Oct. 1986,pp. 547 - 553

Competition Corner

This column is devoted to the Power Question of the 1990 ARML Competition. It was passed along, as always, by Harry D. Ruderman who remarks that Gil Kessler and Larry Zimmerman should be credited for constructing the question.

After your group has solved this very challenging series of problems, try constructing some of your own in the same spirit. Send them [preferably with a solution] to the Editor who will be glad to publish them.

In the following equations, all letters represent positive integers, and a > b.

Let us examine the expression $a^3 + b^3$, where a > b. One well-known result is that $a^3 + b^3 = c^3$ has no solution in positive integers. For each of the equations in parts I and II, either (1) prove that no solutions can exist, or (2) show how an infinite number of solutions can be generated.

I. A. $a^3 + b^3 = c^2$

B. $a^3 + b^3 = cde$, where c,d,and e are in geometric progression

C. $a^3 + b^3 = cde$, where c,d,and e are in arithmetic progression

D. $a^3 + b^3 = 3p$, where p is a prime greater than 3

II. A. $a^3 + b^3 = 2^c$

B. $a^3 + b^3 = 3^c$

C. $a^3 + b^3 = p^c$, where p is a prime greater than 3

III. Assuming that $a^3 + b^3 = c!$ has solutions, and c is at least 12,

A. Prove that the largest prime less than c does not divide a.

B. Prove that a + b is a multiple of 330.

The official solutions will appear in a future issue.

"BUILD BOXES" from page 1

```
LIST
10 INPUT XL,XU,XS
20 DEF FN F(X) = X*(12 - X)*(12 - X)
25 PRINT "HEIGHT", "VOLUME"
30 FOR X = XL TO XU STEP XS
50 PRINT X, FN F(X)
60 NEXT X

 RUN
 ? 6,7,.1

 HEIGHT         VOLUME
  6              216
  6.1            212.341
  6.2            208.568
  6.3            204.687
  6.4            200.704
  6.5            196.625
  6.599999       192.456
  6.699999       188.203
  6.799999       183.872
  6.899999       179.469
  6.999999       175
 Ok
 RUN
 ? 6.35,6.45,.01

 HEIGHT         VOLUME
  6.35           202.7079
  6.36           202.3091
  6.37           201.9092
  6.380001       201.5084
  6.390001       201.1067
  6.400001       200.704
  6.410001       200.3003
  6.420002       199.8956
  6.430002       199.49
  6.440002       199.0835
```

Figure 3

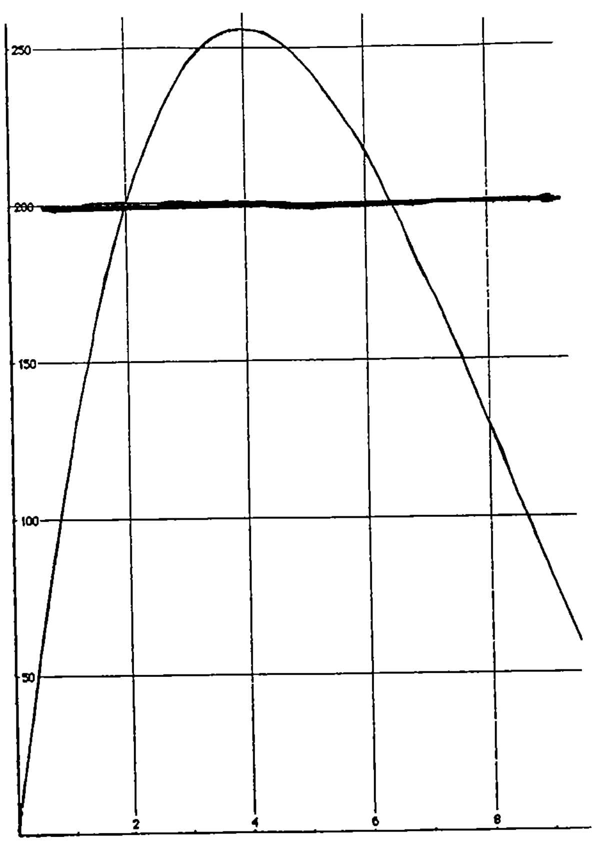


Figure 4

SOLUTION 3

Use a graphing utility [graphing calculator or computer graphing program] to make several graphs of $y = x(12 - x)^2$. [Figures 4,5]

Using the TRACE and ZOOM options, the solutions are a) V= 200 when x = 2 and when $x \approx 6.42$. The maximum value of V is 256 for a box with dimensions 8x8x4.

You could also use an equation-solving program for part a. [Figure 6 shows a solution using TK-Solver]

SOLUTION 4

Again $V = x(12 - x)^2$. The arithmetic-geometric mean inequality for three positive numbers a,b,c states

$$\sqrt[3]{abc} \le \frac{a+b+c}{3}$$ with equality occuring if and only if

$a = b = c$. Choosing $a = x, b = c = 12 - x$ will not quite work since $a + b + c$ is not constant. However letting $a = 2x, b = c = 12 - x$, so that $a + b + c = 24$ and $abc = 2V$,

gives $\sqrt[3]{2V} \le \frac{24}{3}$ or $V \le 256$ for all values of x. Thus the maximum value of V is 256 which occurs when $a = b = c$, or $x = 12 - x$, or $x = 4$. The other dimensions are both $12 - x = 8$.

SOLUTION 5

Take the first derivative of $V(x) = x(12 - x)^2$ to obtain $V'(x) = (12 - x)(12 - 3x)$. Thus $V'(x) = 0$ when $x = 12$ or $x = 4$. $V''(4) < 0$, so the maximum value of $V(x)$ on the interval [0,12] is $V(4) = 256$ for a box with dimensions 8x8x4.

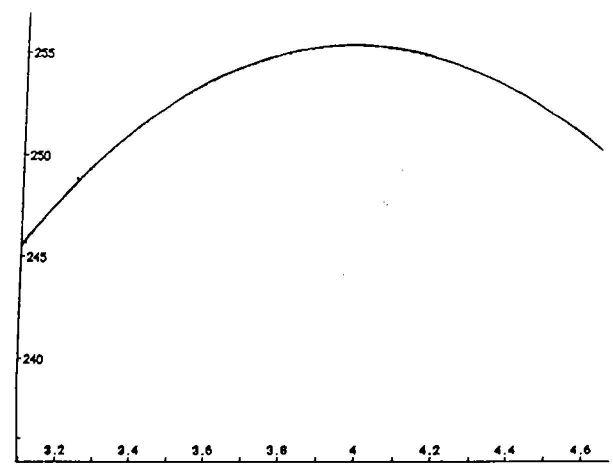


Figure 5

```
VOLUME = LENGTH*WIDTH*HEIGHT
HEIGHT = X
LENGTH = 12 - X
WIDTH = 12 - X
```

HEIGHT	2
X	2
LENGTH	10
WIDTH	10
200 VOLUME	

HEIGHT	6.4174243
X	6.4174243
LENGTH	5.5825757
WIDTH	5.5825757
200 VOLUME	

Figure 6

CONTINUED ON PAGE 6

"BUILD BOXES" from page 5

Commentary on the solutions

Problem 1a: In all five of the solutions, the answer [x = 2] is exact since V(2) = 200, and the other is approximated using a guess-and-test strategy [a root-finding algorithm is a type of guess-and-test method].

Problem 1b: In Solutions 4 and 5 we are certain that V is maximized when x = 4, but in the first three solutions, x = 4 is our best approximation for the value at which V is maximized.

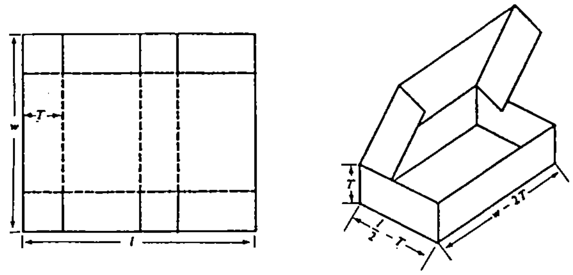
The necessary background for each solution is also of interest. All solutions require the formulation of the algebraic representation for the volume in terms of the height, x, or an equivalent formula. Solution 1 could be determined by a person with a basic knowledge of algebraic expressions and whole number arithmetic. The ability to write a computer program is needed in the second solution while the ability to use computer programs or a graphing calculator [programs written by others] is the key to the third solution. The knowledge of a (perhaps unfamiliar) theorem [Arithmetic-Geometric Mean Inequality] is required in Solution 4. Solution 5 is a "bread-and'butter" calculus solution using the knowledge of several calculus concepts. Thus an 8th grader might determine the first solution while the last one is probably available only to high school seniors or college students.

Which solutions are "acceptable" to you?

Which of these solutions are "acceptable" to you? This issue might make an interesting discussion topic at your next MAΘ meeting.

Try to solve these two classical problems using each of the five methods.

Problem 1 Using the same 24 by 12 sheet of heavy paper, build a "pizza box" as indicated in Figure 7. Find the dimensions of the box(es) whose volume
a. equals 105 b. is maximized.
c. Can you generalize your solution to part b?



Problem 2 A (right circular) cylindrical can is to hold 100 cm^3. Find the radius and height of all cans whose surface area a. equals 200 cm^2 b. is minimized.
c. Can you generalize your solution to part b?

dia Log ue

with Log Editor Tom Butts

This column is empty in this issue because it is intended as a forum for **your** questions and comments. So if you have a question about any area of mathematics - a person, a concept, a problem,... a comment about any of the articles in this issue, or anything else on your mind, write it up and send it to me.[Use the address on page 2.]

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If you're not sure of the significance of this title, look it up.

****

If your favorite dessert is cherry pi, then complete each sentence with an appropriate mathematical term.

1. The zoid hunter hoped to __________ .
2. I'm not against farm implements, in fact, I'm____ .
3. Eyeglasses are good for ______________ .
4. This _________ board has three layers.
5. "__________ ", said the woman urging her parrot to listen carefully to the ex-president.

****

If "quad" usually means four, why does a quadratic polynomial have degree two [and not four]?

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References
1. NCTM Standards, pp. 131-136.
2. "To Build a Better Box", K. Dundas, College Mathematics Journal, 1984, pp. 30 - 36.