

The Mathematical Log

Volume 34 Number 4, December 1990

Data
 AB = 8
 BC = 8.5
 CD = 4
 DA = 2.83

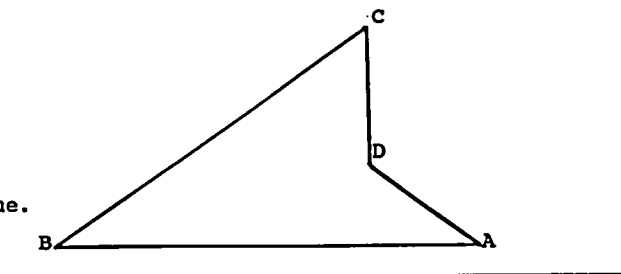


Figure 1

Cannot measure area.
 Press any key to continue.

The Geometric Supposer © gives the message "cannot compute area" for the quadrilateral in Figure 1. By entering its vertices in counterclockwise order, other programs will compute its area. In this article, we will consider some ideas and problems concerning the area of polygons.

The area of a polygonal region is a positive real number that describes its size. For a rectangle, this number is postulated to be the product of two adjacent sides. If a region R can be partitioned into a number of smaller regions R_1, R_2, \dots, R_n ,

then $\text{area } R = \text{area } R_1 + \text{area } R_2 + \dots + \text{area } R_n$. The area formulas of other familiar polygons can be found by relating them to rectangles as shown in Figure 2.

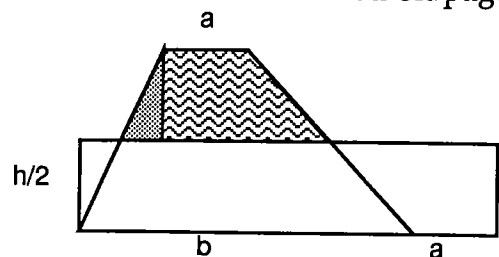
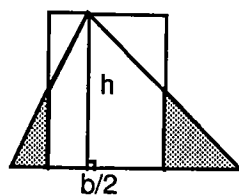
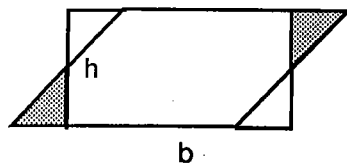
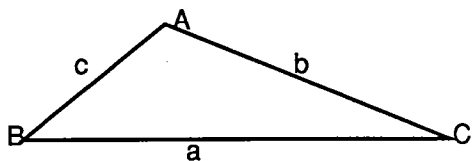


Figure 2

Analyzing Area

Problem 1 Use similar triangles [or some other method] to show the area of a triangle does not depend on which side is chosen as the base. If a, b, c are the sides of a triangle with corresponding heights h_a, h_b, h_c , then

$$\text{the area, } K, \text{ is } K = \frac{1}{2} a h_a = \frac{1}{2} b h_b = \frac{1}{2} c h_c.$$

TRIANGLES

There is an area formula corresponding to each of the three common triangle congruence conditions. For SAS, the familiar area formula given in Fig. 2 can be expressed for a given $\triangle ABC$ as

$$K = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B.$$

Problem 2 Find an area formula corresponding to the ASA condition; i.e. express the area, K , of $\triangle ABC$ in terms of B, C , and a , for example.

The area formula corresponding to the SSS condition is regarded as one of the great theorems of early mathematics. Although some scholars attribute it to Archimedes, it is usually credited to Heron because of the "wondrous piece of abstract reasoning" he used in his proof. It is

Continued on page 5

1990 $\sqrt{\text{At the Root of It All}}$

Deborah Patonai Phillips, Activities Editor

Indispensable attributes for MA Θ sponsors include commitment, dedication, and a whole lot of energy. Two of its most enthusiastic sponsors have recently been elected to the national governing board.

$\sqrt{\text{At the Root of It All}}$ is pleased to honor President-elect Carol McGill and Region Three Governor Frank Caballero.

Currently teaching at Sunset High School in Miami, Florida and at Miami-Dade Community College, Frank Caballero has been teaching mathematics in Dade County since 1974. He communicates a love of mathematics whether teaching geometry, trigonometry, or abstract linear algebra.

With a B.S. degree in Chemical Engineering from the University of Washington in Seattle and an M.A. degree in Mathematics from the University of North Carolina, he has been active in various local, state, and national organizations. He holds memberships in AI, ChE, MAA, NCTM, FCTM, and DDCTM in which he served as vice-president. He was also a delegate to the National Assembly on Standards at the NCTM national convention in Orlando and to a state-wide workshop on competitions.

Frank has been honored with several distinguished teaching awards. In both 1986 and 1990, he received the Dade County School Board Award for Outstanding Contributions to Mathematics Education. In 1986 he was honored with the DCCTM Award for Excellence in Teaching Mathematics as well as the prestigious state Presidential Award for Excellence in Mathematics Teaching.

Involved in MA Θ since 1978, Frank sponsors a successful and active local chapter. They divide their time and efforts into three categories: tutoring, competitions, and FUN! Included in the school-wide tutoring program, members spend at least one hour each day tutoring after school. At the junior high school, they devote one day per week helping 8th and 9th graders. They practice daily for competitions and hold car washes and candy sales to raise money for travel expenses. In January, 1991, they will host a regional math competition for schools from three counties. For the past six years, they have done very well and hope that "1991 will be a banner year". The social highlight of the year is a massive picnic. Inviting former MA Θ members, the chapter take weeks to plan the party with plenty of food, games, and music.

In addition, Frank contributes time to the state organization. Previously serving as vice-president and president, he has been instrumental in creating a strong state MA Θ . He was chairman of the state convention in 1982 and co-chaired the 1988 North Miami Beach/Miami Sunset Invitational Math Competition.

Our new president-elect is Dr. Carol Ann McGill from West Orange-Stark High School in Orange, Texas. Always possessing a desire to teach, Carol earned degrees in English, Mathematics, and Guidance and Counseling from the University of Houston. She has chaired the mathematics department since 1975.

Involved in numerous professional organizations, she has been a speaker at various conventions, has held several offices in the Texas State Teachers Association, and is a member of NCTM, the Texas Classroom Teachers Association, the Sabine Area Council of Teachers of Mathematics, and the University Interscholastic League sponsors. In 1989, she was nominated for the Texas Presidential Award.

Carol's interest in MA Θ began early in the 1980's. By 1983, she was Assistant Editor of *The Log* and wrote the column "The Problem Corner". Later she was named NCTM liaison and this year she was elected to the highest office.

Mu Alpha Theta is indeed fortunate to have two people with such credentials to guide it. Congratulations and good luck to President Carol McGill and Governor Frank Caballero.

The Mathematical Log

Volume 34 Number 4, December 1990

The *Mathematical Log* is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Founded in 1957 by Richard and Josephine Andree, Mu Alpha Theta is co-sponsored by the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM).

Correspondence may be directed to the Editor: Thomas R. Butts, University of Texas at Dallas, P.O. Box 830688 Richardson, TX 75083-0688, or to Mu Alpha Theta National Office, 601 Elm Ave., Room 423, Norman, OK 73019. © 1990 Mu Alpha Theta

Gauss and the Regular Heptodecagon

Ali R. Amir-Moéz, Texas Tech University 79409

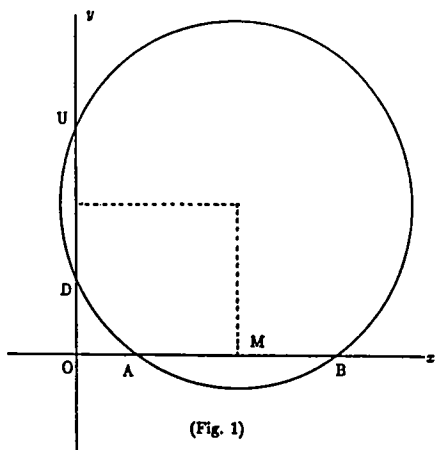
Karl Friedrich Gauss(1777-1855) at the age of eighteen gave a ruler-compass construction of dividing a circle into seventeen equal arcs.

1 Quadratic Equations:

Let

$$ax^2 + bx + c = 0, \quad a \neq 0 \quad (1)$$

be a quadratic equation with real coefficients. Consider a rectangular coordinate system (Fig. 1). Let A and B, on the x-axis, correspond to x_1 and x_2 , respectively, where x_1 and x_2 are the roots of (1). Let $OU = 1$ and $OD = \frac{c}{a}$. Then



(Fig. 1)

$$(OU)(OD) = \frac{c}{a} = x_1 x_2 = (OA)(OB). \quad (2)$$

This implies that the four points, U, D, A, and B are on circle [1]. Since M, the midpoint of AB, corresponds to

$$\frac{1}{2}(x_1 + x_2) = \frac{-b}{2a}, \quad (3)$$

we can construct the circle which passes through U, D, A, and B. Therefore the roots of (1) can be constructed.

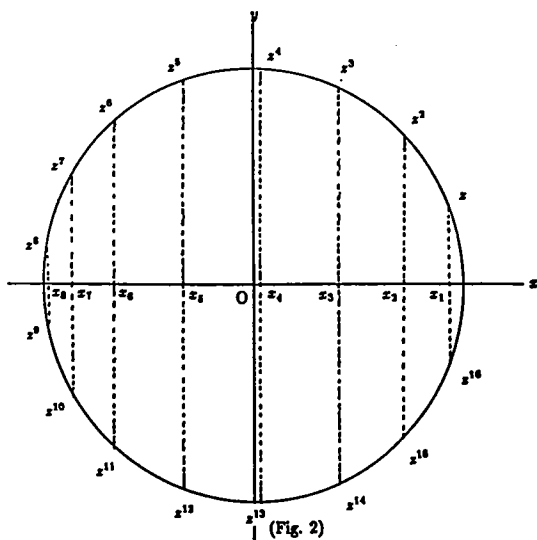
2 The Seventeenth Roots of Unity:

Consider the equation

$$z^{17} - 1 = 0. \quad (4)$$

One can easily factor (4) into

$$(z - 1)(z^{16} + z^{15} + \dots + z^2 + z + 1) = 0. \quad (5)$$



(Fig. 2)

So one root of (4) is $z = 1$ which is a real number. The other roots are complex and can be written as

$$z_k = \cos \frac{2\pi + 2k\pi}{17} + i \sin \frac{2\pi + 2k\pi}{17}, \quad k = 1, \dots, 16. \quad (6)$$

Let $z_0 = 1$ and $z_1 = z$ (Fig. 2). Then we observe that

$$z_k = z^k, \quad k = 1, \dots, 16. \quad (7)$$

We also note that

$$z_k = z_{17-k}^*, \quad k = 9, \dots, 16. \quad (8)$$

where, for example, z^* is the conjugate of z . This implies that

$$z_h + z_{17-h}^* = 2 \cos \frac{2(\pi + h\pi)}{17}. \quad (9)$$

If we let

$$z_h = x_h + iy_h, \quad (10)$$

then

$$z_h + z_{17-h}^* = 2x_h, \quad h = 9, \dots, 16. \quad (11)$$

One can write (11) as:

$$z_k + z_{17-k}^* = 2x_k, \quad k = 1, \dots, 8. \quad (12)$$

3 The Ruler-Compass Construction:

The multiplicative group of residue classes of integers (mod 17) has sixteen elements (order 16). Lagrange's theorem suggests the subgroup

$$\{1, 9, 13, 15, 16, 8, 4, 2, \} \quad (13)$$

and the corresponding coset

$$\{3, 10, 5, 11, 14, 7, 12, 6\}. \quad (14)$$

So Gauss has considered

$$\begin{aligned} \ell_1 &= \frac{1}{2} \{ (z + z^{16}) + (z^2 + z^{15}) + (z^4 + z^{13}) + (z^8 + z^9) \} \\ &= x_1 + x_2 + x_4 + x_8. \end{aligned} \quad (15)$$

and

$$\begin{aligned} \ell_2 &= \frac{1}{2} \{ (z^3 + z^{14}) + (z^4 + z^{12}) + (z^6 + z^{11}) + (z^7 + z^{10}) \} \\ &= x_3 + x_5 + x_6 + x_7. \end{aligned} \quad (16)$$

One may easily show that $\ell_1 > 0$ and $\ell_2 < 0$. Now we observe that

$$\ell_1 + \ell_2 = -\frac{1}{2}, \quad \ell_1 \ell_2 = -1 \quad (17)$$

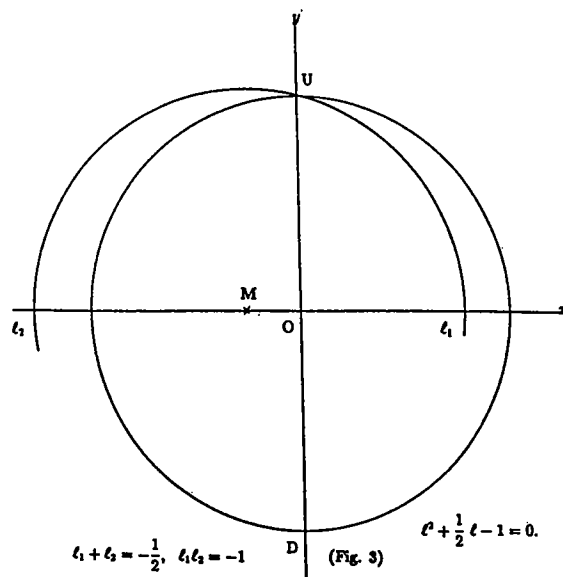
Thus ℓ_1 and ℓ_2 satisfy the quadratic equation

$$\ell^2 + \frac{1}{2}\ell - 1 = 0. \quad (18)$$

One can construct the roots of (18) (Fig.3). Comparing with (Fig. 1), we note that

$$OD = -1 \text{ and } OM = -\frac{1}{4}. \quad (19)$$

So ℓ_1 and ℓ_2 are obtained as in (Fig. 3).



(Fig. 3)

Next we observe that if we let

$$m_1 = \frac{1}{2} [(z + z^{16}) + (z^4 + z^{13})] = x_1 + x_4 > 0, \tag{20}$$

and

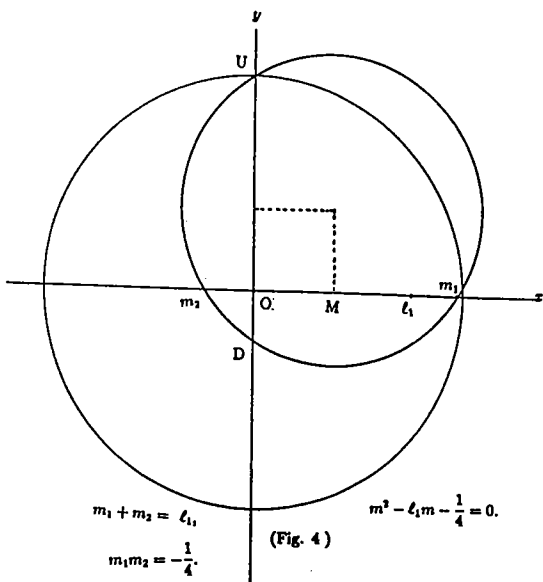
$$m_2 = \frac{1}{2} [(z^2 + z^{16}) + (z^8 + z^9)] = x_2 + x_8, \tag{21}$$

we get

$$m_1 + m_2 = x_1 + x_2 + x_4 + x_8 = \ell_1, \tag{22}$$

and

$$m_1 m_2 = -\frac{1}{4}. \tag{23}$$



So m_1 and m_2 satisfy the quadratic equation

$$m^2 - \ell_1 m - \frac{1}{4} = 0. \tag{24}$$

By carrying ℓ_1 from (Fig. 3), we can construct m_1 and m_2 (Fig. 4). The reader may examine the figure against Figures 1 and 3.

Now we let

$$n_1 = \frac{1}{2} [(z^3 + z^{14}) + (z^5 + z^{12})] = x_3 + x_5, \tag{25}$$

and

$$n_2 = \frac{1}{2} [(z^6 + z^{11}) + (z^7 + z^{10})] = x_6 + x_7. \tag{26}$$

Note that

$$n_1 + n_2 = x_3 + x_5 + x_6 + x_7 = \ell_2, \text{ and } n_1 n_2 = -\frac{1}{4}. \tag{27}$$

Therefore n_1 and n_2 satisfy the quadratic equation

$$n^2 - \ell_2 n - \frac{1}{4} = 0. \tag{28}$$

Constructing the roots of (28) we get (Fig. 5).

Finally, we note that

$$x_1 = \frac{1}{2}(z + z^{16}) \text{ and } x_4 = \frac{1}{2}(z^4 + z^{13}). \tag{29}$$

So

$$x_1 + x_4 = \frac{1}{2}(z + z^4 + z^{13} + z^{16}) = m_1, \tag{30}$$

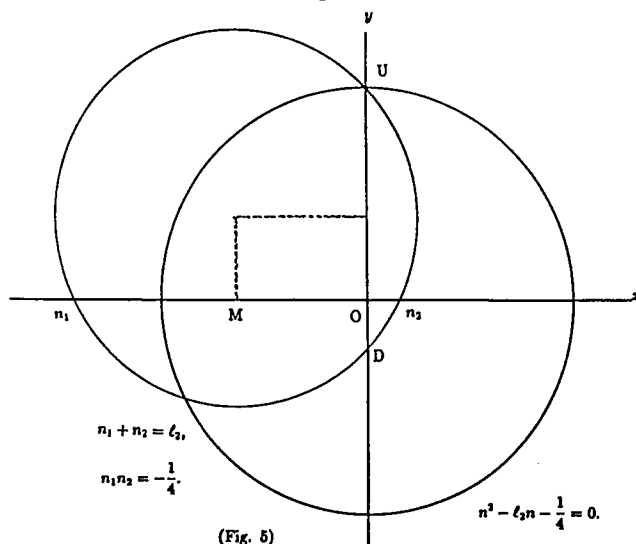
and

$$x_1 x_4 = \frac{1}{4}(z^5 + z^{14} + z^3 + z^{12}) = \frac{1}{4} n_1. \tag{31}$$

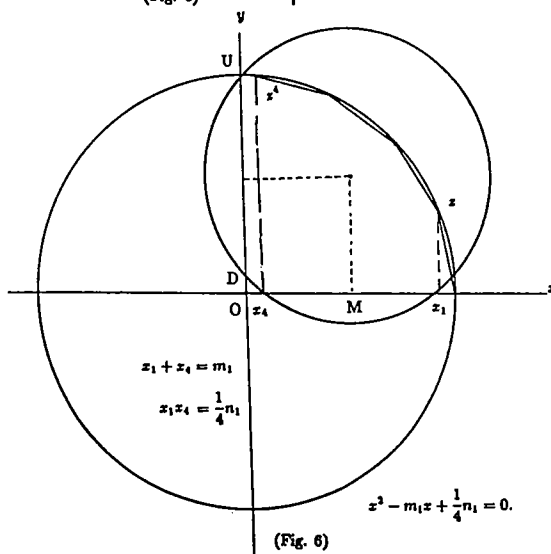
Therefore x_1 and x_4 satisfy the quadratic equation

$$x^2 - m_1 x + \frac{1}{4} n_1 = 0. \tag{32}$$

Construction the roots of (32) we obtain (Fig. 6). From x_1 and x_4 we get z and z^4 . The rest of the construction is quite clear.



(Fig. 5)



(Fig. 6)

Time adds up fast for math professor

Lubbock - Too many interests - so little time.

Seventy-one years haven't been long enough for Ali R. Amir-Moez to explore all the fascinations he has with life as a mathematician, artist, writer, actor, and dancer.

The mathematics professor emeritus at Texas Tech University, who came to the United States from Iran in 1947, has experienced life from the logical and scientific to the artistic and aesthetic. "But having so many interests has often made life complicated," he says.

He believes, "What you do is worthless if it does not help people. I try to help mankind with the things they do."

He has created math-oriented puzzles and games for children, has written more than 200 articles and books in numerous areas of higher mathematics, and has guided the research of doctoral students.

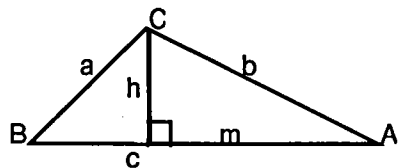
For several years he taught children ages 9 - 12 in Tech's summer program for gifted students. "I have always been interested in children's education and math learning," Amir-Moez said. "I have found that children are bright." The best compliment of his teaching career came from a 12-year old boy who asked him, "Why can't we come on Saturday, too?"

Area ... from page 1

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. For Heron's proof, see [1].

Problem 3 Prove Heron's Formula using the steps:



(1) $m = \frac{b^2 + c^2 - a^2}{2c}$ (2) Substitute m into

$h = \sqrt{b^2 - m^2}$; (3) Substitute h into $K = \frac{ch}{2}$.

QUADRILATERALS

An early [incorrect] Egyptian formula for the area of a quadrilateral with sides a,b,c,d is $K = \frac{a+c}{2} \frac{b+d}{2}$ [the product of the averages of its opposite sides.]

Problem 4

- a. Show $K \leq \frac{ad+bc}{2}$. When does equality occur?
- b. Show $K \leq \frac{a+c}{2} \frac{b+d}{2}$. When does equality occur?

Since a quadrilateral is not determined by its sides, it is plausible that a general formula for its area also involves some of its angles. In fact,

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \left(\frac{A+C}{2} \right)$$

where s is the semiperimeter. For a cyclic formula of quadrilateral, this reduces to the celebrated of Brahmagupta,

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Problem 5

- a. In what way is Heron's formula a special case of Brahmagupta's formula?
- b. If a cyclic quadrilateral also has a circumscribed circle, how can Brahmagupta's formula be restated?

Problem 6 Derive these formulas.

Some other interesting area formulas for quadrilaterals are contained in the next few problems.

Problem 7

- a. The area of a convex quadrilateral with diagonals d_1 and d_2 forming $\angle\theta$ is given by $K = \frac{1}{2} d_1 d_2 \sin \theta$.
- b. What is the special case of this formula for a rhombus or a kite?

Problem 8 A convex quadrilateral is partitioned into four triangles by its diagonals. For what type of quadrilaterals do

- a. exactly two of the non-adjacent triangles have equal area?
- b. all four triangles have equal areas?

Problem 9 Why do you think the area of the quadrilateral in Fig. 1 could not be computed directly by the Geometric Supposer ®? How could you use the Supposer to compute this area?

Problem 10 Find the area of a trapezoid with bases a and b, $a > b$, and acute angles A and B between the longer base and the non-parallel sides.

COORDINATE AREA FORMULAS

For a polygon whose vertices are at "nails" on a geoboard or, equivalently, at lattice points in the coordinate plane [points (x,y) where both x and y are integers], there is an interesting area formula due to Gyorgy Pick [1899]. The area, K, can be expressed in terms of I, the number of lattice points ["nails"] in the interior of the polygon, and B, the number of lattice points on its boundary.

Problem 11 See if you can rediscover this formula by considering several cases when $I = 0$, then when $I = 1$, then when $I = 2$, etc.

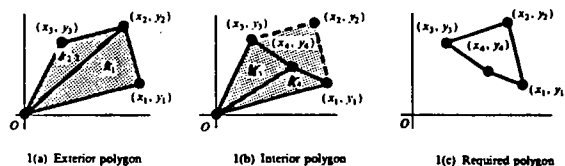
The area of a triangle whose vertices are $(0,0)$, (x_1, y_1) , and

(x_2, y_2) in counterclockwise order is given by $K = \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

This formula can be generalized to find the area of any polygon given the coordinates of its vertices. The area of the quadrilateral in the figure below can be expressed in terms of the areas of the four triangles by $K = K_1 + K_2 - (K_3 + K_4)$. Using the basic formula

[being sure vertices are listed in counterclockwise order] we have

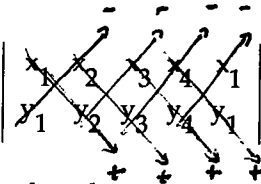
$$K = \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} x_4 & x_3 \\ y_4 & y_3 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix}$$



Mu Alpha Theta T-shirts are here! They are blue and come in sizes M, L, and XL. Order yours today. Send \$8 for each T-shirt to the national office.

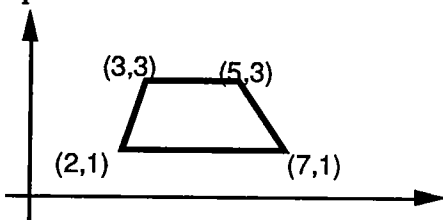
$$\text{or } K = \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix}$$

This can conveniently be rewritten as

$$K = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix} \text{ if we interpret } \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$


by summing products on the "downward" diagonals and subtracting the products on the "upward" diagonals.

Example



The trapezoid has area

$$K = \frac{1}{2} \begin{vmatrix} 2 & 7 & 5 & 3 & 2 \\ 1 & 1 & 3 & 3 & 1 \end{vmatrix} = \frac{1}{2} [2 + 21 + 15 + 3 - 7 - 5 - 9 - 6] = 7.$$

This can be confirmed using the usual formula for the area of a trapezoid.

This formula can be generalized to any n-gon, whether or not it is convex, given the vertices in counterclockwise order.

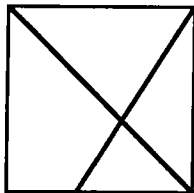
Problem 12 Write this general formula and make up an example to check it.

Problem 13 Confirm that the area of the quadrilateral in Fig. 1, whose vertices are (1,1), (9,1), (7,3), and (7,7), is 20 in three ways:

- using the polygonal coordinate area formula
- using Pick's Theorem
- partitioning the quadrilateral into two triangles.

Solve each of these last two problems in several ways. Find a generalization if you can.

Problem 14



- The area of the square is 36. A segment from one vertex to the midpoint of the opposite side and the

diagonal between the other two vertices are drawn. Find the areas of the four regions formed.

- Repeat part a if the original quadrilateral is a parallelogram of area 60.

Problem 15

Let P, Q, R, and S be the midpoints of the sides AB, BC, CD, and DA respectively of convex quadrilateral ABCD. Find the ratio area PQRS: area ABCD.

- Dunham, *Journey Through Genius*, Wiley, 1990, Ch. 5

Log niappe

If your favorite dessert is cherry pi, then complete each sentence with an appropriate mathematical term.

- Her name was placed on the ballot by the ____.
- This priceless book is a first ____.
- A book about submarines is a _____.
- I took a _____ to the Vatican.
- When I tell my landlord to come and collect his money every month, I say "_____".

• Gauss requested that a heptodecagon be engraved on his tombstone. Although the request was not honored, such a polygon is found at the base of a monument to him at his birthplace in Brunswick [Germany].

• Gauss' solution to the construction of the heptodecagon was one of the first to employ the problem solving strategy of "moving" a problem from one subject [geometry] to another [algebra].

Last issue answers: 1. trapezoid 2. protractor 3. division 4. multiply 5. polyhedron. For 'quadratic', consult an unabridged dictionary to see the connection with quadrature.

OOPS! Joan Reinthaler is the author of the Mathematics and Music: Some Intersections. Giving the correct spelling of her name gives me another opportunity to encourage you to purchase this wonderful monograph for \$2.50 from the national office.

The deadline for the GEOBOGGLE Contest has been extended to February 15, 1991. Send in your entry today.