**Seattle Convention / Back to School Issue**

# The Mathematical Log

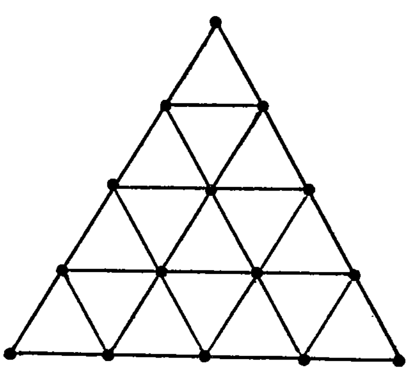

**Volume 31, Number 3** **October 1987**

## Interesting Geometric Activity

# Parallelograms on Equilateral Triangles Yield Geometric Counting Problem

By David R. Duncan and Bonnie H. Detwiller

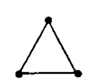
An interesting geometric activity involves counting various types of geometric objects. One such activity involves counting the total number of parallelograms (including rhombi) that can be placed on an n by n by n equilateral triangle, as shown (for n = 4) in Figure 1.



4 by 4 by 4 Triangle
Figure 1

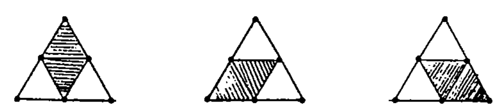
Count the parallelograms whose vertices lie on the lattice points of the figure and whose sides lie on the "lattice lines." To organize the counting, start with simple cases.

Case 1: 1 by 1 by 1 triangle:



There are 0 parallelograms.

Case 2: 2 by 2 by 2 triangle:



Only a 1 x 1 parallelogram (rhombus) can be positioned on the triangle; however, it may be positioned in three ways. Thus, there are 3 parallelograms.

Case 3: 3 by 3 by 3 triangle:

There are 9 - 1 by 1 parallelograms and 4 - 2 by 1 parallelograms. The 2 by 1 parallelograms are shown in Figure 2.

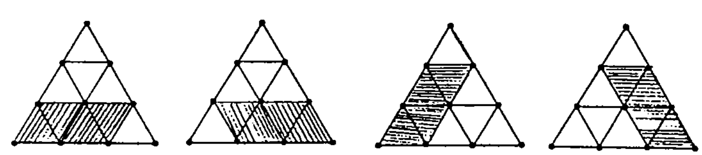


Figure 2
2 by 1 Parallelograms

(Concluded on Page Two)

# Parallelograms on Equilateral Triangles

... FROM PAGE ONE

TABLE 1: NUMBERS OF PARALLELOGRAMS ON n BY n BY n EQUILATERAL TRIANGLES, n = 1 THROUGH 7

| Size of Triangle | 1 by 1 | 2 by 1 | 2 by 2 | 3 by 1 | 3 by 2 | 3 by 3 | 4 by 1 | 4 by 2 | 4 by 3 | 4 by 4 | 5 by 1 | 5 by 2 | 6 by 1 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 by 1 by 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 by 2 by 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 3 by 3 by 3 | 9 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 13 |
| 4 by 4 by 4 | 18 | 12 | 3 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 37 |
| 5 by 5 by 5 | 30 | 24 | 9 | 12 | 4 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 83 |
| 6 by 6 by 6 | 45 | 40 | 18 | 24 | 12 | 3 | 12 | 4 | 0 | 0 | 4 | 0 | 0 | 162 |
| 7 by 7 by 7 | 63 | 60 | 30 | 40 | 24 | 9 | 24 | 12 | 4 | 0 | 12 | 4 | 4 | 286 |

Case 4: 4 by 4 by 4 triangle:

There are: 18 - 1 by 1 parallelograms
12 - 2 by 1 parallelograms
3 - 2 by 2 parallelograms
4 - 3 by 1 parallelograms

The 2 by 2 parallelograms are shown in Figure 3. Table 1 gives the results for n by n by n triangles, where $1 \leq n \leq 7$.

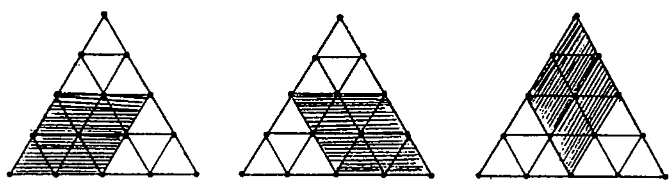


Figure 3
2 by 2 Parallelograms

Several patterns may be observed.

1. The first time a rhombus "fits" in a triangle, it occurs in three positions.
2. Beginning with the first occurrence of a rhombus, the numbers of such rhombi in successive triangles are 3 times the triangular numbers. Recall that the triangular numbers are 1, 3, 6, 10, 15, 21, ....
3. The first time a parallelogram which is not a rhombus "fits" on a triangle, it occurs in four positions.

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David R. Duncan and Bonnie H. Litwiller are Professors of Mathematics--and well-known contributors of ideas for the math classroom--at University of Northern Iowa, Cedar Falls, IA 50614.

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4. Beginning with the first occurrence of a parallelogram which is not a rhombus, the numbers of such parallelograms are 4 times the triangular numbers.

Challenges to the readers:

1. Extend Table 1 for 8 by 8 by 8, 9 by 9 by 9, and 10 by 10 by 10 triangles.
2. Determine the formula for the first occurrence of a parallelogram of any given size.
3. Express the relationships of Observations 2 and 4 using recursive relations.

[Also, reaching for matchsticks and glue, consider whether something interesting happens when the whole problem is "elevated," by analogy, to a higher dimension ... parallelopipeds in an n by n by n by n tetrahedron? ... lattice-point figures in hyperspace? --ED.]

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The Mathematical Log is the official publication of Mu Alpha Theta, national high school and junior college mathematics honor society and mathematics club federation. Mu Alpha Theta, founded in 1957 by Richard and Josephine Andree, is co-sponsored by the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics. The Mathematical Log is published four times a year, in February, April, October, and December. Correspondence may be addressed to specific editors or to Mu Alpha Theta National Office, 601 Elm Ave., Rm. 423, Norman, OK 73019. 

# Millipedes, Arrowheads Pose 'Life' Challenges

Millipedes "die," arrowheads become "doughnut" pairs, and a "tilted six-pack" oscillates--we think! Louisiana member Todd Belton, who shared "Life" pattern investigations in December's Mathematical Log, has sent us his notes on further "Life" discoveries, inviting reader help with the "why" if not the "what" of the generations unfolding on his notepad or computer screen.

English mathematician John Conway's captivating "Life" game/activity has been thoroughly chronicled by mathematics popularizer Martin Gardner, and its "basics" are spelled out in the April 1985 Mathematical Log.

Todd's "millipedes" and "arrowheads" (he calls them "arrows") are novel "Life" configurations which he invented and has been investigating. One charm of "Life" is that it does lend itself to such original consideration.

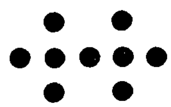

Figure 1

Todd's "basic millipede" is the nine-dot, four-"leg" configuration illustrated below (Figure 1). One "segment" longer, it becomes the thirteen-dot, six-"leg" creature of Figure 2. Logical extension is to Todd's "very long

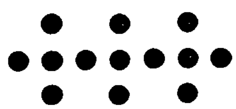

Figure 2

millipede" (Figure 3). "All seem doomed to an unhappy death," Todd notes. But, "why?"

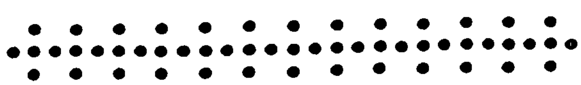

Figure 3

Todd's "arrowheads," however many, stabilize as a pair of "doughnuts"--a well-known "Life" stable form. The doughnuts are in about the position shown (Figure 4), the graphics being based on Todd's working sketch. A seven-

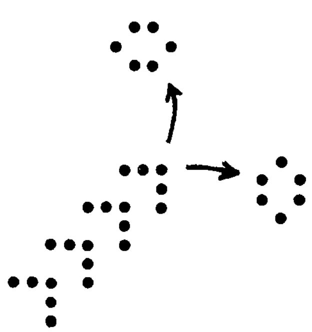

Figure 4

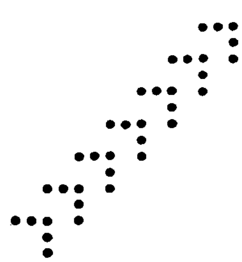

Figure 5

"arrow" configuration is shown in Figure 5. Its "Life" outcome is "the same two doughnuts: try it yourself, you won't believe it till you do," Todd writes.

"Why does it happen? Can it be proven?" Todd asks.

Todd's "tilted six-pack" (our name for it!) is the "Lifeform" depicted in Figure 6. "In the course of investication of [what Gardner calls] 'gliders,' I encountered this lifeform. Try it!" Todd writes. We have,

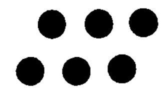

Figure 6

and get a repeated "two-cycle," a "blinker" osscilation. The reader may wish to check our results. H.D.A.

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Todd Belton's home address is 11622 Pamela, Baton Rouge, LA 70815. He welcomes "Life" correspondence.

# Superb Questions Highlight Latest California Contest

(Good contest questions are what most Log-readers read first! This we know from watching Log distribution at national conventions. And, for good contest questions, few competitions come close to the Santa Clara University annual high school mathematics contest. With the generous permission of Santa Clara's Gerald L. Alexanderson, reproduced below is the full November 1986 question set. --Ed.)

1. If the notation $a^{b^c}$ means $a^{(b^c)}$, list the following numbers in descending order, i.e., from largest to smallest:

$$2^{3^4},\ 4^{2^3},\ 3^{2^4},\ 4^{3^2},\ 2^{4^3},\ 3^{4^2}.$$

2. Write the following sum as the product of two nontrivial factors (i.e., neither factor is 1 or the number itself):

$$27^8 + 125^8 + 15^{12}.$$

3. It is easy to show that $2^2 - 1$ and $2^3 - 1$ are factors of $2^6 - 1$, since $2^6 - 1 = (2^3)^2 - 1$. Similarly $2^{10} - 1$ has both $2^2 - 1$ and $2^5 - 1$ as factors.
   - (a) If $2^6 - 1 = (2^2 - 1)(2^3 - 1)A$, express A as a sum of powers of 2 and hence find the binary representation of A.
   - (b) If $2^{10} - 1 = (2^2 - 1)(2^5 - 1)B$, find the binary representation of B.
   - (c) If $2^{14} - 1 = (2^2 - 1)(2^7 - 1)C$, find the binary representation of C.
   - (d) If $2^{18} - 1 = (2^2 - 1)(2^9 - 1)D$, find the binary representation of D.

4. Four circles of equal radius $r$ are placed around a given circle of radius $b$, so that each of the four circles is tangent to the given circle and tangent to each other successively. Find the radius $r$ of the four circles in terms of the radius $b$ of the given circle.

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On Problem Solving

"The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc., is the solution of mathematical problems. It is the duty of all teachers and of teachers of mathematics in particular, to expose their students to problems much more than to facts.
--Paul R. Halmos, quoted in College Mathematics Journal.

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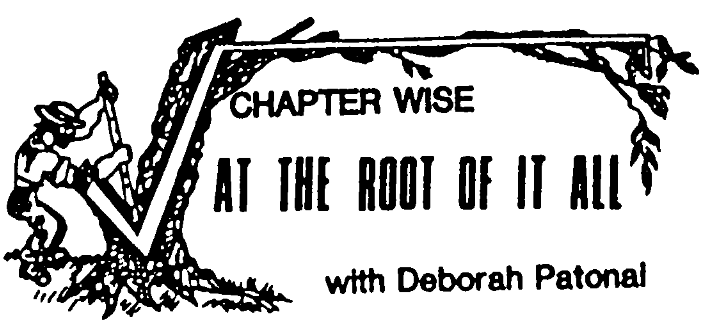
5. Find the length of the side of a regular 24 sided polygon inscribed in a circle of radius $a$.

6. Find 5 prime factors of $2^{90} - 1$.

7. If we proceed $\frac{5}{12}$ of the way along a diagonal of a cube of edge length 1 (the diagonal connects opposite vertices of the cube), then slice the cube with a plane perpendicular to that diagonal,
   - (a) What is the shape of the polygon formed by the intersection of the plane and the cube?
   - (b) What is the perimeter of the polygon?

MATHEMATICS IS EVERYWHERE, these Summer School students find, in the numerosity of buckets of golf balls at a driving range (left), in the swirls of a freshly cut Mobius band. Others, rightly, view mathematics in the combinatorics of license plates, the probabilities of bridge hands, the randomness of winning lottery numbers. Nature, too, abounds in mathematical allusions, from the spirals of sea shells and spider webs, the normal distributions of animal and plant populations, the patterns in leaf clusters or daisy petals. Geometry manifests itself in symmetries in life forms, crystal structures, frost tracings. Those who are comfortable with the language and concepts of mathematics are best able to view and to appreciate their world.




A small but important percentage of Mu Alpha Theta chapters are those thriving at junior colleges. Totalling less than twenty chapters nationwide, these junior college groups continue and extend work done by high school chapters of Mu Alpha Theta. Promoting greater interest in mathematics and giving recognition for excellence in mathematics performance are goals common to high school and junior college chapters. One such active but young junior college chapter of Mu Alpha Theta can be found on the campus of Allegany Community College in Cumberland, MD.

Behind every successful Mu Alpha Theta chapter is an influential sponsor. The Allegany Community College chapter has such a sponsor in Gary Shaffer, assistant professor of mathematics. Not new to Mu Alpha Theta, Professor Shaffer had been a secondary school sponsor at Bedford High School in Pennsylvania. Soon after he left Bedford to teach at Allegany in 1983, he initiated the process of establishing a college chapter of Mu Alpha Theta.

During Spring Semester of 1985, Allegany Community College received its Mu Alpha Theta charter. Thirteen students, possessing qualifications which were in compliance with guidlines suggested in the Mu Alpha Theta Handbook for Sponsors as well as those agreed upon by the Allegany Mathematics Department, were chosen as charter members. Believing that at junior college level Mu Alpha Theta is best regarded as a mathematics honor society and not solely a "club," the College paid the $2 individual fee from an activity fund. Membership in Mu Alpha Theta, accordingly, is an honor bestowed upon Allegany students! Stressing the honorary aspect of Mu Alpha Theta enables this chapter to draw a budget from the student activities fund, thus eliminating the need for fundraisers.

Holding two meetings per semester, this junior college chapter seeks a diversity of exercises and activities for members. One meeting may be devoted to working out problems from a Mathematical Log; another to discussing papers or features in Mathematical Buds, Mathematics to Play and Ponder, or the Andree Cryptarithm publications. One unusual session involved tying of rope to both wrists of two members so that the "closed curves" formed by each person's arms-rope-body were interlocking "circles." The members then tried to get apart without untying the ropes. The chapter is considering providing this "challenge" in connection with week-long, 25th anniversary celebrations of the College.

In addition to meetings for its own members, the Allegany chapter has become involved in a number of college-wide activities. It has established a mathematics tutoring service. A student member has been chosen to serve on the student government of the College. The Club has been asked for their input and their reactions to a proposed "honors" curriculum that the College plans for the Fall of 1987. The chapter sponsors and presents the Mu Alpha Theta Outstanding Mathematics Student Award at the all-college awards banquet for graduating sophomores. The group also participates in the AMATYC Mathematics League contest for junior colleges.

Working with Allegany's math department, this Mu Alpha Theta chapter co-sponsors an annual tri-state area high school mathematics contest, held on the Allegany campus. Mu Alpha Theta members help in formulating contest questions, proctor the contest, and participate in welcoming remarks and awards presentations. The College Foundation provides scholarship money for the top ten finishers, area businesses supply other prizes, and Mu Alpha Theta donates the winning team plaque.

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REFLECTING ON MATHEMATICS LEARNING

As I look back upon my own experience I find that the best lessons of life were the hardest. ... My mother's insistence upon daily exercises in mental arithmetic has been worth more to me than all the delightful dallyings with intellectual pleasures I have ever had. Life is not a pastime, and democracy is not a holiday excursion.

--Charles Evans Hughes,
Former Chief Justice of the United States Supreme Court.

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As the Allegany sponsor, Gary Shaffer is very positive about the future of Mu Alpha Theta on the Maryland campus. In 1990 the chapter will be five years old. He already is planning an alumni dinner to renew acquaintances of former Mu Alpha Theta students.

Gary Shaffer would like to see more two-year colleges become involved in Mu Alpha Theta.

He believes:

"Participation in Mu Alpha Theta is another building block helping to establish a firm mathematics foundation at any institution. It is an excellent way to interact with students who have an interest in mathematics outside the classroom.

"With Mu Alpha Theta the potential is there for whatever the imagination can conceive."

Mu Alpha Theta, clearly, has found a good home on the campus of Allegany Community College.

# Richard V. Andree, 1919 - 1987

# SHARING PROBLEMS "JOY - PRODUCING" TO MU ALPHA THETA FOUNDER

1. The triangle with sides 9, 10, 17 has the property that its perimeter is [numerically] equal to its area. (Don't take our word for it; compute it and see for yourself.) Determine other triangles having all three sides measured in integers and whose area and perimeter are equal.

2. Let A, B, C, D, E be five towns such that the straight line distance in miles between them is

| AB | BC | CD | DE | EA |
|---|---|---|---|---|
| 30 | 80 | 236 | 86 | 40 |

Find the distance between C and E without a map.

3. What are the final three digits of

$$X = (((\ \dots\ (((7)^7)^7)^7\ \dots)^7)^7,$$

where there are 1000 exponents, each of which is 7?

4. Take a rectangle of 13 by 22 units. Your problem is to cover it completely with smaller rectangles (<u>total</u> area 13 x 22 = 286 square units) having the property that the smaller rectangles also have <u>integral lengths of sides</u> and such that <u>no covering rectangle can fit inside any other of the covering rectangles</u>. Consider a 4 x 6 rectangle fits inside a 4 x 7 rectangle (or even another 4 x 6 rectangle), but a 4 x 6 and a 3 x 7 rectangle do <u>not</u> fit inside one another.

* * *

Dr. Richard V. Andree, who with his wife (and fellow mathematician) Josephine Peet Andree founded Mu Alpha Theta in 1957, died on May 8 in Norman, OK, where he had taught university mathematics and computer science for 37 years. Dick Andree was well known to thousands within and beyond Mu Alpha Theta for his many and diverse publications, his <u>Mathematical Log</u> contributions and other problem-solving materials, his work with young people and his genial presence at conventions. At the time of his death at 67, which followed an extended illness, Dick Andree was Professor Emeritus at the University of Oklahoma, Norman, where he had chaired the mathematics and astronomy department from 1961 to 1969.

Dick Andree served as National Secretary-Treasurer (1957-78) and National President (1978-81) for Pi Mu Epsilon, a national honorary society in mathematics for undergraduate majors and graduate students. He was a Fellow of the American Association for the Advancement of Science and recipient of the Regents Award for Distinguished Teaching, the Mu Alpha Theta Distinguished Service Award, and the Oklahoma Association for Gifted and Talented Award. Dick's "influence on our department and in mathematics in our state and nation was and is very significant," colleagues noted at the time of his death. The most recent public recognition of Dick's unique service to mathematics and mathematics education was the naming of the Richard V. Andree Future Teacher Award (April <u>Mathematical Log</u>, p. 1), the first winner of which is announced in this issue.

Dick Andree's textbooks and other writings, including those coauthored with his wife or wife and son, introduced a generation of teachers and others to important mathematical areas. His <u>Modern Abstract Algebra</u> was the <u>Log</u> editor's late-1950's introduction to concepts underlying significant curriculum reform.

In a society with oral rather than written traditions, Mu Alpha Theta's Dick Andree very likely would be remembered as a unique, even legendary, individual who stimulated the thinking and challenged the minds of youth. His fascination was with open-ended problems, non-routine problems, the creative side of problem posing and problem solving. His knack for harnessing the computer to extend perspectives and broaden horizons, set patterns for others to follow. In a society with written traditions, where all can read and quite a few do, Dick Andree and his thinking live on through the problems he created and so freely shared. Generations can, and perhaps will, share in the challenges and the good fun, glimpsing something of the "joy-producing" aspect (Dick's apt phrase) of a good problem well posed and resourcefully "opened up."

We began this <u>Log</u> tribute to Dick and his work with a cluster of the good problems that Dick Andree shared with us one convention, for <u>Mathematical Log</u> and <u>Tall Timbers</u> inclusion "when you need something to run" (see <u>Timbers</u> 1 and 2). We conclude with a further grouping. Do give them a try. There is much that they can teach you now, even as they linger to tantalize and to teach you and others for generations to come.

* * *

5. **Begin with a positive integer $N_0$. As an example take $N_0 = 47831$**

Let $N_1$ denote the sum of the squares of the digits of $N_0$. $N_1 = 139$

Let $N_2$ denote the sum of the squares of the digits of $N_1$. $N_2 = 91$

Let $N_3$ denote the sum of the squares of the digits of $N_2$. $N_3 = 82$

. $N_4 = 68$

. $N_5 = 100$

. $N_6 = 1$

Let $N_{k+1}$ denote the sum of the squares of the digits of $N_k$. $N_7 = 1$

. . . $N_k = 1$

Thus, $N_0 = 47831$ produces the sequence at the right. On the other hand, $N_0 = 1783$ produces:
{1783, 123, 14, 17, 50, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, 89},
which loops back onto the series [note the repetition of the 89] forever.

Investigate this for other positive starting values $N_0$. Make a conjecture and prove or disprove your conjecture.

6. Find two rational numbers the sum of whose cubes is six. (This problem, which "baffled A. M. Legendre, one of the world's greatest mathematicians" (ac-

(Concluded on Page Six)

## Future Teacher Award

# Love of Subject Matter Interest in People Suggest Career

Mu Alpha Theta convention audiences--and individual members--frequently are urged to consider careers in mathematics teaching. Each student would have his or her own reasons for arriving at such a career decision. When Matt Frueh, Mu Alpha Theta's first Andree Prospective Teacher Award winner, was asked his reasons for choosing mathematics, the following was his response.

By Matt Frueh

The great English poet, Geoffrey Chaucer, once wrote that "full wise is he that can himselven know." After many hours of self-examination (and countless cups of acidic coffee), I have succeeded in isolating what I believe to be my candid reasons for wanting to become a mathematics teacher. In the process I hope that I have not only reconfirmed my desire to teach but also gained some measure of self-knowledge and wisdom as well.

Chief among my reasons for wanting to teach mathematics is my love of the subject matter itself. I find mathematics extremely fascinating and challenging; in all honesty, it is something I never tire of. C[arl] F[riedrich] Gauss is credited with terming mathematics "the queen of the sciences." I believe this otherwise exceptional man was wrong in one respect--he should have said "king."

Personal love of the subject, however, is not enough. I also want to help other people to enjoy mathematics--no easy task, given the stigma traditionally attached to it; math, like spinach, is something one is "supposed" to hate. The "misguided" few who actually find mathematics intriguing are labeled "nerds" or "geeks," particularly, but not exclusively, at the high school level. As I see it, the degree to which a student is interested in mathematics varies directly with his or her teacher's dedication and competence. I'm convinced that I possess the dedication, and will attain the competence, necessary to help make the student mathematician more than just the infrequent oddity that he or she is today.

Finally, I choose the teaching profession because it provides the kind of people-centered environment not found in most other fields. For several years I was certain that I wanted to be a computer programmer; I was fascinated as much by the high-tech glamour surrounding the occupation as I was by the job itself. After some thoughtful reconsideration, however, I decided that a full-fledged career centered about computers might give rise to certain unwanted side effects, namely the kind of burnout and alienation that results from what I consider to be a stale, unrewarding atmosphere lacking in human interaction. I still like computers, and in fact have chosen Computer Science as my college minor. But the satisfaction derived from the teaching process would, I think, far surpass anything (including the higher salary) obtainable from writing programs.

In conclusion, let me say that I sincerely believe I am entering the mathematics teaching profession for the right reasons and with sound objectives. To do otherwise would mock not only Mu Alpha Theta and the Andrees, but also my educators, parents, and everyone else who has consistently supported my choice of occupation. I look forward to the day when I, as a teacher, will be able to nominate one of my own students for an award such as this.




REAL-WORLD MATH takes many forms, as these student leaders learned "on location." Allan Matheson, New Waterford, NS (left) and Russell Pierce, Margaretsville, NS, took the Editor and their college-division chapter to the high-technology world of the Government of Canada Regional Emergency Government Headquarters in Debert, NS, a fifties-style fallout shelter on immense scale. Students found the unusual excursion of distinct interest for the math and physical science questions and dialogue it evoked. Many chapters arrange for visits to business and industry, to universities, or to the military as part of a full, all-year agenda.

# SHARING PROBLEMS

... FROM PAGE FIVE

cording to Dick Andree), leads the student to worthwhile procedures and extensions: "Like all good explorations one success uncovers other summits to conquer." The question lends itself to computer investigation: "Modern tools make it possible to do many things that were impossible a century or two ago," Dick has noted. The two rationals can be seen on careful scrutiny of the current Tall Timbers.)

7. Find all values of N less than $10^9$ such that $N^2$ ends in N.
8. Write a computer program to accept three values, M, D, Y, and determine whether or not they are acceptable values for Month, Day, Year, in that order. If Y is less than 1753 you may prefer to state the date is unacceptable rather than consider the various calendar reforms that took place prior to 1753. Note: 2000 will be a leap year, but 1900 was not.
9. How many primes of the form $X^2 + 1$ can you find? For example, $2^2 + 1 = 5$ is prime.
10. The equation $X^3 + Y^3 + Z^3 = 3$ has solutions (1, 1, 1) and (4, 4, -5). Determine as many additional integer solutions as you can.
11. Find all five-digit positive integers that contain exactly the same digits as their trebles. The leading digit must not be zero. Example: 10035 * 3 = 30105.
12. Either find a 10-digit positive integer N with each digit different and such that N is also a perfect 10th power of an integer, or else show that no such N exists.

--H. D. A.