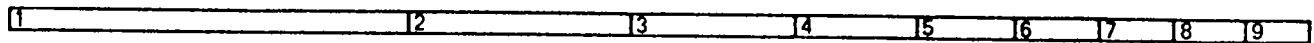




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'Research' Area

THREE 'MEANS' OF SCHOOL MATH GEOMETRICALLY REPRESENTED

By Ali R. Amir-Moéz
Mathematics Editor

Means of positive real numbers are useful and interesting. In this note we study them from the point of view of geometry.

1. Means: Let a, b be two positive real numbers.

I. The arithmetic mean of a and b is:

$$m = \frac{1}{2}(a + b);$$

II. The geometric mean of a and b is:

$$g = \sqrt{ab};$$

III. If we let h denote the harmonic mean of a and b , then we have:

$$\frac{2}{h} = \frac{1}{a} + \frac{1}{b}.$$

Note that for any two numbers one may define the arithmetic mean, but for other means one needs some restrictions.

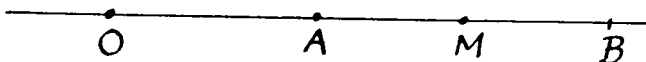


Fig. 1

2. Geometric construction of means: Let us choose an axis (Fig. 1). We construct $OA = a$ and $OB = b$. Then $m =$

$\frac{1}{2}(a + b)$ corresponds to M , the midpoint of the line segment AB .

Now we shall repeat the first drawing and add the circle of center M and radius MA (Fig. 2). Let OT be the tangent to the circle. We choose $OG = OT$ on the axis. One can easily show that $OG = g = \sqrt{ab}$. The triangle OMT has a

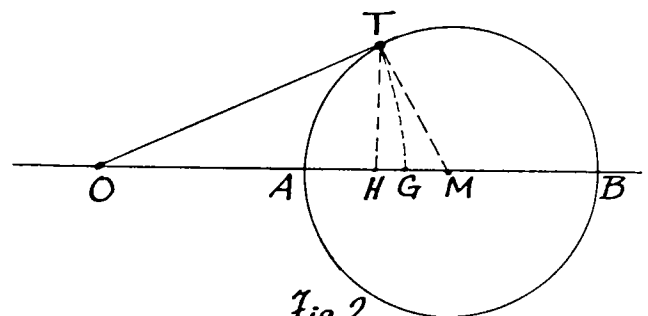


Fig. 2

right angle at the vertex T . Thus,

$$OT^2 = OM^2 - MT^2.$$

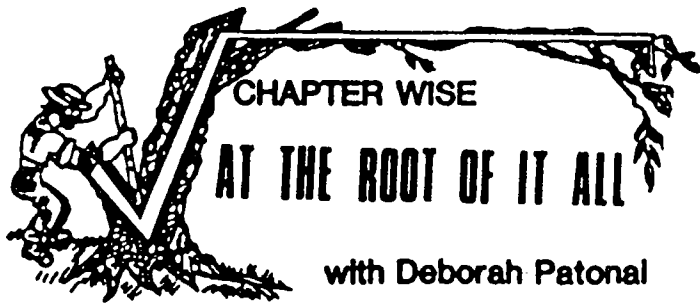
One observes that

$$MT = \frac{1}{2}|a - b|, \quad OM = \frac{1}{2}(a + b).$$

Thus,

$$OG^2 = OT^2 = \frac{1}{4}[(a + b)^2 - (a - b)^2] = ab.$$

Now we draw the perpendicular TH to the axis (Fig. 2). (See "Three 'Means,'" page 8)



with Deborah Patonai

Behind every successful Mu Alpha Theta chapter is a dynamic, impelling faculty sponsor who leads the club to success. The sponsor should not only demonstrate organizational and leadership skills but also express the caring concern of a mathematics teacher. $\sqrt{\text{At the Root of It All}}$ has once again uncovered such a sponsor. She is Claudia Carter, Mu Alpha Theta faculty advisor at St. Mary's Dominican High School, New Orleans, LA.

The first ingredient of an effective Mu Alpha Theta advisor is to be an exceptional mathematics teacher who can excite students. Claudia Carter is this type of stimulating educator. As a Mu Alpha Theta sponsor, Claudia can enjoy the challenge of working with the brighter students. However, as a math teacher she encounters a variety of students. One not-so-motivated group of students is always asking the question, "Why do we have to learn this stuff?"

Being a problem solver, Claudia decided to tackle this question. Believing "an appreciation of mathematics evolves from a practical understanding of the history of mathematics," Claudia realized students in all levels lack knowledge about the origins of mathematics. Instead of what students labelled as "boring books about boring mathematicians," Claudia wanted to put together something the students could relate to and enjoy. In 1984 Claudia published an historical novel of mathematics, Challenge Through Time.

Challenge Through Time (Charles Gifford, publisher) makes the history of mathematics come alive. One's imagination is sparked by the adventures of four teenagers who journey back into time to learn about the origins of mathematics. According to Claudia, the book "provides a stimulus for historical excerpts found in textbooks, as a prerequisite for more complex histories or related projects, for enrichment, or for reinforcement." Accordingly, "it can demonstrate to students that mathematics is an ever-growing subject."

A second ingredient of a valuable Mu Alpha Theta sponsor is to become extremely involved. Being busy with her own Mu Alpha Theta chapter was not enough for Claudia. She wanted her Dominican chapter to become involved at Mu Alpha Theta levels outside her school. She decided to run students for state offices of president and vice-president. After an exciting election both Dominican girls won. Too late Claudia discovered that, as their sponsor, she would be responsible for running the State convention the following year! Much to her credit, the State convention, which hosted over 750 students, sponsors, and speakers, was a huge success.

A third ingredient of a notable Mu Alpha Theta sponsor is "to be a little bit crazy"--in Harold Huneke's apt turn of phrase. Shortly after the Louisiana state convention, New Orleans was nominated for Mu Alpha Theta's 1984 National Convention. When the chairperson was unable to fill the commitment, Claudia was approached by several sponsors to take charge. "No" is not in Claudia's vocabulary! Making Mu Alpha Theta history, the 1984 Convention attracted over 900, another tribute to Claudia Carter and her committee of sponsors from New Orleans.

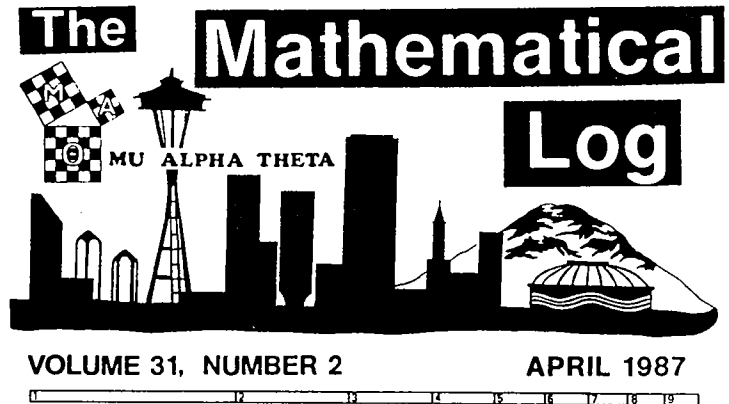
Since then, Claudia has chaired another Louisiana state convention, and has served on the Louisiana Mu Alpha Theta executive board.

Being an avid national Mu Alpha Theta convention person since 1980, Claudia can be seen coaching her students

to success--or dancing the night away, as she did in Miami.

While Claudia Carter believes "Mu Alpha Theta is a truly '+' experience for us all," I believe she is a "+" for Mu Alpha Theta!

NATIONAL CONVENTION, UNIVERSITY OF SEATTLE, AUGUST 1ST-5TH
MU ALPHA THETA'S "EVENT OF THE YEAR" ... MAKE PLANS NOW!



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80 AREAS

Projects, Presentations Can Reflect Range Of Math Interests

By Don Allen

Since becoming a Mu Alpha Theta sponsor twenty-five years ago, we've fielded our share and more of good math questions ... but two favorites always will be:

"What might make a good program for next meeting?"

"What could I do that's different and interesting for a Math entry in the Science Fair?"

The two questions are not unrelated, when you think about it. A strong Math Fair entry should be a natural for the Math Club agenda. And many an imaginative program can provide the impetus for a thrust of individual or small-group research.

Accordingly, when local teachers recently approached us for Science Fair "math suggestions," we put on our Mu Alpha Theta thinking cap, and looked to good programs and sound areas for individual investigation. We came up with eighty, plus an extended bibliography. The listing of eighty "topic areas" follows. The bibliography is running in our current (April 1987) Mathematical Tall Timbers chapter supplement.

Numerical entries of two digits are cross-references to related areas. Three-digit references are to specific bibliographic listings.

Sponsors and students are cordially invited to share their own suggestions, for additional topics or perspectives on topics and for good "sources" in support of topic areas.

* * *

01. Patterns and patternlessness I. Numbers: multiples, progressions, other regular sequences, primes, variants of primes [11], other number patterns [68], depicted on rectangular grids (rows, diagonals, the Ulam "spiral" [Dudley, 070]), triangular, hexagonal, or other grids. Arithmetic or algebraic interpretation of geometric patterns perceived. Randomness correspondingly depicted ("chance sequences," winning lottery numbers) [52].

02. Patterns and patternlessness II. Space: "random walks." Graphing sequences of "moves" as vectors in one, two, three dimensions. Depicting, considering random sequences (from coin flips, dice tosses, "real life," computer RND function, etc.) Predicting, verifying longer-term outcomes. Traditional "drunkard's walk" investigations. Game-board variants. Relation to gambling "runs," to Brownian movements [52]. Random elements in art [71].

03. Real-world mathematics, then and there. Numeration (words, writing instruments and surfaces, symbols, associated conventions), computation (operations, algorithms, mechanical and other aids), utilization (commerce, trades, astronomy), in other times, places. Observations as to persistence of additive, multiplicative, subtractive principles, existence of zero, concepts of base and place value. Babylonian (cuneiform, sexagesimal), Egyptian (counting numbers, unit fractions [028, 171]), Iconic Greek and Roman ("modern" Roman, as "evolved"), systems

of the Hebrew heritage, China and the Orient, pre-Columbian America (Aztec, Mayan), others. [Standard histories of mathematics, encyclopedias, 008, 027, 044, 080, 111, 237, 251, 252.]

04. Numbers as shapes; a touch of antiquity. Pythagorean perspective of numbers as shapes (square, rectangular, pyramidal). Illustrating, investigating, relating figurate and similar number patterns in one, two, three dimensions [005]. Models. Identifying, demonstrating figurate identities. Associated sequences. General expressions. [022, 047, 162, 210, 214, 252, 322]

05. Evolution of mathematical notation--and more. An historical overview, with contemporary "illustrations," of development of number concepts, number notation (fractions, surds, decimals, exponents), operation and relation signs and conventions, symbols for functions, variables, constants (π , e). Relevance of notation to understanding, concept development. Origins of signs, conventions, now in universal use. [252 and similar compilations, 047, 080, and Gardner's 111.]

06. Older ways. Insight into mathematical processes from early algorithms (Russian peasant multiplication [273, 293]) or from topics no longer current (arithmetic cube root). Logarithmic computation [traditional textbooks, 210]. Slide-rule computation. Nomographs [066]. Computational/manipulative expectations of early courses/texts/examinations and the competencies they assumed (arithmetic, algebra, geometrical construction, trigonometry).

07. Mathematics that might have been. Conjecturing, defining, exploring, extending other possible approaches to numeration; other operations, other assumptions. Positive integral number bases greater than or less than ten. Fractional bases. Negative bases. Factorial and prime bases. Counting, computing in such systems. Divisibility criteria, repetend length, modifications to "carrying" and "borrowing." Checks. Models and demonstrations relating to such systems. [040, 111, 273]

08. Modelling binary arithmetic. Number representation (integers, rationals, irrationals) using binary notation. Binary counting. Carrying out of fundamental operations. Representation of zero/one as off/on (light), open/closed (switch). Computer relevance. Adaptation as octal, hexadecimal.

09. Investigating mathematical conjectures: the counting numbers. Form, test, refine, possibly prove or disprove, original conjectures or conjectures encountered in reading and discussion, relating to such classes of counting numbers as even and odd numbers, square and triangular numbers, prime and composite numbers, "twin primes," perfect numbers. Present, interpret findings. Typical conjectures: powers of odd numbers are odd, no square has 8 as digital root, one or both of $6n \pm 1$ invariably is prime (DeBouvelles' conjecture, 1509), every even number greater than 6 can be written as the sum of two distinct primes (Goldbach's conjecture, 1742, "strong-
(See "Projects, Presentations," page 4)

Projects, Presentations

... FROM PAGE THREE

er version"). [See introductory Number Theory texts, especially 070; also, 031, 165, 176.]

10. Factoring techniques for some larger integers. Demonstration, application of divisibility tests, and establishment, recognition of "factorable forms" (e.g., sums of cubes) in prime factorization of larger integers. Computer procedures. Review of the literature of known large primes, partial and complete factorizations. Relevance to cryptography. [272, 152]

11. Varying the sieve. Discussion, graphical demonstration of classic "sieve of Eratosthenes" as efficient means of eliminating multiples, obtaining extended listing of initial prime numbers. Exploration of possible alternative "sifting" techniques, and of prime-like sequences resulting from their application [004]. Formulation, testing, refinement of conjectures concerning one or more such number types, possibly conjectures analogous to those for prime numbers and "twin primes" [10, 070].

12. Rational numbers, repetends, irrational numbers, and infinite, non-repeating expressions. Distinction among "wholly repeating," "partially repeating," "terminating" decimal expressions. Relevance of the base. Repetend length. Explorations and conjectures. Irrational expressions. Computational techniques for irrational roots ("traditional," Newton's method, computed-suited approaches). Properties of some resulting non-terminating, non-repeating digit sequences [18]. [Introductory Number Theory textbooks; also, 085, 259, 315.]

PROJECTS AND PRESENTATIONS--CHAPTERS' CHOICE

A good program and a good "fair" project have much in common--planning, action, investigation, communication! Yet a good program can tap a wider dimension: a tour, a visiting lecturer, a broad look at mathematics and mathematicians in the real world. Do share your best projects and programs--in response to this feature and a "sharing" component of your Mu Alpha Theta way of life. News of chapter activities is prominently featured in your Log--whenever such news is received. --Ed.

13. Solution in integers. Diophantine equation defined. Particular solution of the linear Diophantine equation. Existence of solutions. General solution. Application to problem solving. Relation to continued fractions [14]. Historical notes. [Histories of mathematics, some Number Theory textbooks.]

14. Simple continued fractions. Finite, repeating infinite, and nonrepeating infinite continued fractions and their identification with rationals, quadratic surd expressions, other irrationals. Simple continued fraction convergents as increasingly good rational approximations to final rational value, quadratic surd, or other irrational such as pi. Historical notes. Relevance to problem solving. [Old "Higher Arithmetics"; some Number Theory textbooks, with 269 outstanding.]

15. Figures called "magic." Squares, cubes, circles, star polygons, and other such configurations have been deemed "magic" when their cells or points have number values associated with them (typically consecutive integers) which add or otherwise combine to a "magic constant." Magic squares in particular have been known since antiquity, and an enormous literature exists on their construction and extension. An historical presentation might combine with demonstration, algebraic justification of construction techniques, possibly an attempt at innovative extrapolation [233, 076, 097, 110, 141, 201, 303, 304, 321].

16. Looking to "differences." Observing, computing, conjecturing concerning successive "differences," patterns of differences, for terms of such sequences as linear, quadratic, other polynomial, rational, exponential,

factorial--and such special sequences as prime numbers [9], Fibonacci numbers [68]. Investigating, demonstrating, applying a "calculus of finite differences" [281]. Interpolation, extrapolation. Determining functions giving rise to stipulated "difference patterns." Newton's formula, other "difference relations." [180, 235]

17. More than asked for: extraneous roots and "solutions extraneous to the problem." "Extra" solutions tend to be summarily dismissed in school mathematics, but their implications do not so easily go away! A detailed consideration of how extraneous roots arise in equation solution and of "answers" which do not satisfy conditions of the underlying problem. Relation to fallacies [70]. Implications for problem solving.

18. Presenting and approximating pi. Computing the mathematical constant pi to increasing precision using approaches ranging from measurement of circles, ellipses, cylinders, cones, spheres, to perimeter, area computation for inscribed, circumscribed regular polygons of increasing numbers of sides. Also, sequence evaluation (anticipating Calculus), statistical approaches (Buffon's needles). Decimal, common fraction, continued fraction [14] expressions. Historical notes. Application, demonstration in mensuration: spherical surface area, volume. [026, 062, 134]

19. Functions and functionality. Definitions, modern and historical. Characteristics of a function, graph of a function. "Function machines." Some "special functions" (greatest integer, absolute value, exponential, logarithmic, circular, hyperbolic). Graphing of functions. Functionality in interesting, real-life contexts.

20. Pythagoras revisited. The Pythagorean theorem and its converse lend themselves to an exceptional range of proofs and demonstrations, offering insight into the diversity of possible approaches. Locate the Loomis classic [243]. Further, generalization or extension of the Pythagorean theorem--instructively--might be to similar figures on hypotenuse, sides; to "general triangle" consideration (as the cosine law of Plane Trigonometry); or to other surfaces (e.g., spherical) or higher dimensions [046, 082, 178, 205].

21. Elegant Euclidean geometry that Euclid never knew. Extending traditional triangle-circle concepts of Greek geometry through such later developments as the nine-points circle, Euler line, and Simson line. Concepts defined. Properties demonstrated through Euclidean, other proofs. Large, accurate diagrams are a real challenge. [055, 080, 117, 202, 273]

22. Adding a new tool. What happens to geometric constructibility when one augments the "straightedge and compasses" instrument kit of classical Greece? That is, what happens when construction elements no longer are restricted to points, lines and line segments, circles and circular arcs? Devising one or more supplementary "instruments" or procedures and exploring construction possibilities that are found to result. Relation to conic sections, spirals, other curves [30; 022, 127, 229, 273, 327].

23. Modelling in 3-space. Construction, exhibiting of surface and skeletal models of polyhedral and other "mathematical solids" from patterns, faces, rods (straws). Identifying, modelling Platonic, Archimedean solids, duals, stellate and articulated forms. Demonstrating Euler's relation for numbers of polyhedral faces, edges, vertices. Face, edge, vertex coloring experiments. Historical notes. Relation to architecture, geodesic domes. [025, 034]

24. Projections and perspectives. Architectural drawing. Rules of perspective. Historical notes. Generalized consideration of "shadow" of n-dimensional object onto n-1 dimensions. [195, 218, 333]

25. Glimpsing the fourth dimension. The topic has strong appeal. Consider "modelling" the four-dimensional tesseract in three-dimensional perspective (pipe cleaners work well) [141]. Read on, discuss "visualizing" the fourth dimension [137, 203, 332]. Further background reading should add "perspective" [001, 067, 121].

(See "Projects, Presentations," page 5)

Projects, Presentations

... FROM PAGE FOUR

The fourth dimension and the notion of infinity are topics with distinctive attraction for some students.

26. Visual illusions. Geometry, perspective, and tricks our mind plays on us with what we visually perceive. Ambiguous figures [193]. Illusions and perspectives [218]. Necker cube, Schroder stairs, reversible goblet illusions [017]. Relevance to certain work of Escher [277].

27. Anamorphic art. Something of a "lost art" technique where portraiture, scenes, geometric forms are so rendered that they appear "normal" only when viewed in a cylindrical, spherical, or other reflecting/distorting surface. Aspects of the geometry of anamorphosis should be "retrievable" by the student interested in experimenting with perspective and related considerations. Rendering of "math shapes," still life compositions, scenes, could yield a highly distinctive project. Bergamini provides a fine color reproduction [029], Gardner a worthwhile discussion [093]. See also 268.

28. Finite geometries. Such finite mathematical systems as a "25-point geometry" open up attractively and reveal a surprising richness of conceptualization. Develop terminology, assumptions, theorem sequences. Offer demonstrations, conjectures. Consider possibilities for further extension [081, 273].

29. Coordinateization. Coordinate systems for the ray and line, circle (modular), plane (rectangular, oblique, polar, other), 3-space (rectangular, cylindrical, spherical), n-space, surfaces (latitude, longitude on the sphere). Parametrization. Conversions among systems (e.g., rectangular to polar in the plane). Advantages of different approaches.

30. Curves and surfaces. Definitions, constructions, principal properties of line, circle, conics, and a range of significant curves and surfaces additional to those usually treated in school. Spirals, epicycles (observed planetary motion), cycloids (prolate, curvate [114]), the catenary, cardioid, folium, lemniscate, witch [238], helix. Normal curve. Sine curves and their compositions, surfaces of rotation. Applications in nature, design, technology. Historical associations. [060, 065, 273]

31. Symmetries. Showing plane, axial (line), central (point) symmetries through illustrations from natural, man-made environments, and through such demonstrations as paper folding ("snow flake" generalized [32]), ink-blot "art," and formal designing for symmetries (patterns, tessellations [35]). Rigorous definitions. Relevance to Algebra, Coordinate Geometry, Calculus. [027]

32. Paper-folding discovery. A wide range of potential for creativity and for thoughtful investigation. Possibilities extend from elements of origami [090] to Gardner's "flexagons" [091] and combinatoric "folds" [105]. [Look also to 135, 138, 175, 185, 202.]

33. Geodesic (minimum path) considerations: the "spider on the wall" classic and its variants. This basically counterintuitive gem cries out for model construction, clear exposition, attempted generalization/extension. See the Shyers consideration of "reflective paths" [289], Winthrop on "collapsible locations" [324], and the section in Davis and Chinn [062].

34. Aspects of lattice-point geometry. A feeling for "the lattice of integers" [139] might have its start in "taxicab geometry" (see Byrkit [045]). Alternately, systematic investigation might be directed to "geoboard polygons," initially on a 9-peg (3 x 3) board (8 triangles, 16 quadrilaterals, ... [021]). Extension might be to larger grids, different grids, higher dimensions. Pick's theorem.

35. Tessellations, tilings, and repetitive designs. Investigation, possible development or extension, of repetitive patterns that serve to "tile" the plane or fill space. Regular, semiregular tessellations [030, 233]. Sets of polygons that "surround" a point. Quilt patterns, other such traditional uses. Islamic patterns [064].

Mosaics on spherical, other surfaces. Historical notes. [032, 037, 119, 148, 158, 163, 198, 279, 283, 286]

36. Principles and practices yielding Escher-like repetitive art. Familiarization with mathematics-related artworks of Dutch graphic artist M. C. Escher (1898-1972). Escher's preoccupation with polyhedral and related shapes, the Möbius band, perspective and infinite regression, repetitive "tilings." Production of Escher-like repetitive designs by systematic modification of tessellations to yield, typically, life forms. Mathematical principles involved. Demonstration of techniques, to yield finished Escher-like work. [279, 116, 141, 194, 286, 306]

37. Curve stitching. Coordinatizing, initially, intersecting rays or lines and the circle, investigating systematically families of curves obtainable as envelopes of tangent lines when numbered points are connected according to stipulated rules of regular or modular arithmetic. (Representatively, on 72-point circle (1-72), with modulo 73 reduction, joining n to $2n$ can be shown to yield cardioid, n to $3n$ nephroid, n to $4n$ curve with three cusps, "three-leaf clover.") Patterns and progressions of patterns. Effects of varying numbers of points, multipliers, related rules. Computation, execution of a pattern sequence as a crafts project. Extension to third dimension. [033 (theoretical basis), 242]

38. Lissajous, pendulum art, and banknote tracery. Setting up, demonstrating graphical potential of sand or light pendulum. Consideration of Lissajous (Bowditch) figures, other possible renderings of complex periodic motion. Use of oscilloscope. See college physics references, 267.

39. Far-out curves I. A range of unusual curve-forms for development as hand work, presentation, possible extension. Snowflake and cross-stitch curves [178]. Dragon curve and worm curves [092, 133]. Spirolaterals [227, 261, 287].

40. Farther-out curves II. Computer-age extensions of traditional curve, shape and form concepts. "Fractals" and their applications. The Mandelbrot set [329].

41. Biorhythms, Wilhelm Fliess, and mathematical periodicity. Periodic function defined, illustrated. Circular (trigonometric) as periodic functions. Attempts to apply periodicity concepts to identify cycles (emotions, stock prices, sunspots). Possibilities, limitations of the approach. Gardner is instructive [124].

42. Topological "networks" in real-world contexts. Tracing "routes" on timetable schematics of subway lines (London, Paris, New York, Montreal); municipal transit routes; highway networks; air, rail, or bus connections. Asking good questions. Conjectures, demonstrations. Relating classic "route" problems (e.g., Königsberg bridges) to familiar or imaginative contexts. [027, 029, 036, 052]

43. Labyrinths and mazes in two or more dimensions. Mathematically intriguing, the venerable topic of labyrinths and mazes is considered by Gardner [089] and, more briefly, Palmer [266]. The subject should reward library research and practical experimentation [029, 321].

44. Knots and notations. Characteristics, traditions, classifications of knots, braids. Some related topological considerations: puzzles (Loony Loop), activities ("escaping" tied wrists, removing vest without first removing jacket). An algebra of "cat's cradle" [012, 013]. Other extensions and applications [057, 138, 255].

45. "Map coloring" excursions: exotic surfaces and down-to-earth geographic realities. Topological "proper colorings" conjectures in historical perspective. Demonstrations on the plane, sphere, Möbius band, torus (doughnut), pretzel. Chromatic polynomial. Coloring investigations on polyhedral faces, edges, vertices. Correspondences between maps, graphs. Application of "proper coloring" to complex political realities (local counties, American "48 states," nations of South America, modern Africa). [029, 048, 056, 101, 104, 179, 294]

(See "Projects, Presentations," page 6)

