Convention, Back-to-School Issue

The Mathematical Log

Volume 30, Number 3 OCTOBER 1986

'Aesthetically Pleasing' Theorem Relates Triangle Areas

A Solid Geometry theorem characterized as "both aesthetically pleasing and mathematically interesting" has been suggested--and proven by Coordinate Geometry--by a San Antonio high school student. The efforts of David Salony, a junior at Alamo Heights High School, have been submitted to Mu Alpha Theta by the teacher who encouraged David in his work.

Noting that David's demonstration involves only Algebra and Coordinate Geometry--and the corollary Vector Algebra--Alamo Heights teacher Carlynn Ricks has suggested that other students might be interested in the conjecture and the proof.

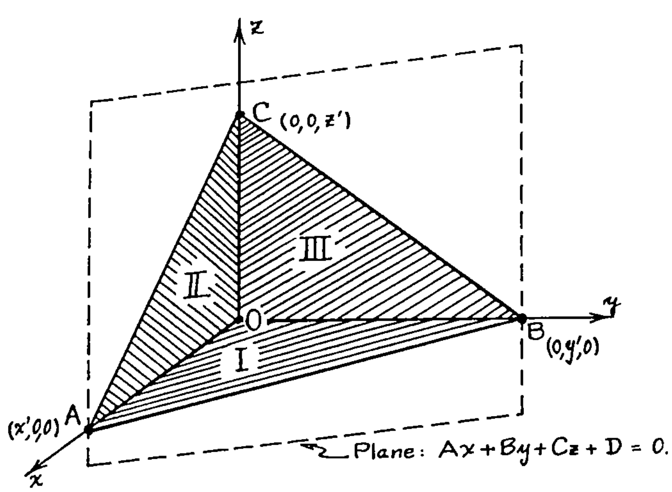
The theorem, proof, and corollary, essentially as submitted to Ms. Ricks, The Mathematical Log has pleasure in sharing. For this publication, for its Mathematical Tall Timbers supplement, and for Mathematical Buds student paper compilations, Mu Alpha Theta welcomes such student insights at any time.

By David Salony

Given: The general equation of a plane in space:

$$Ax + By + Cz + D = 0$$

where neither A, B, nor C is equal to zero.

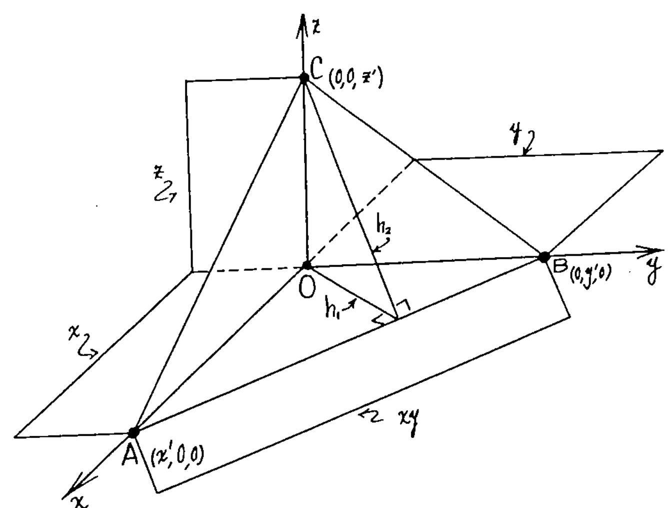


Prove: The square of the area of the triangle formed by the three traces of the plane is equal to the sum of the squares of the areas of the three triangles formed by the traces and the three axes; that is:

$$(\text{Area } \triangle \text{I})^2 + (\text{Area } \triangle \text{II})^2 + (\text{Area } \triangle \text{III})^2 = (\text{Area } \triangle ABC)^2.$$

Proof:

$$\text{Area } \triangle AOB = \tfrac{1}{2}xy$$
$$\text{Area } \triangle AOC = \tfrac{1}{2}xz$$
$$\text{Area } \triangle BOC = \tfrac{1}{2}yz$$



Now,

$$AB = \sqrt{x^2 + y^2}$$

$$\text{Area } \triangle AOB = \tfrac{1}{2}(xy)(h_1), \text{ so}$$

$$\tfrac{1}{2}xy = \tfrac{1}{2}(\sqrt{x^2 + y^2}) \cdot h_1$$

$$h_1 = \frac{xy}{\sqrt{x^2 + y^2}}$$

By the Pythagorean theorem,

$$h_2 = \sqrt{\left(\frac{xy}{\sqrt{x^2 + y^2}}\right)^2 + z^2}$$

$$= \sqrt{\frac{x^2y^2}{x^2 + y^2} + z^2}$$

$$= \sqrt{\frac{x^2y^2 + z^2(x^2 + y^2)}{x^2 + y^2}}$$

$$= \sqrt{\frac{x^2y^2 + x^2z^2 + y^2z^2}{x^2 + y^2}}.$$

(See "Triangle Areas Theorem," p. 2)

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Triangle Areas Theorem

... FROM PAGE ONE

Now,

$$\text{Area } \triangle ABC = \tfrac{1}{2}bh$$
$$= \tfrac{1}{2}(xy)(h_2)$$
$$= \tfrac{1}{2}\sqrt{x^2+y^2}\cdot\sqrt{\frac{x^2y^2+x^2z^2+y^2z^2}{x^2+y^2}}$$
$$= \tfrac{1}{2}\sqrt{x^2y^2+x^2z^2+y^2z^2}.$$

Accordingly, we ask:

$$(\text{Area } \triangle AOB)^2 + (\text{Area } \triangle AOC)^2 + (\text{Area } \triangle BOC)^2$$

$$\stackrel{?}{=} (\text{Area } \triangle ABC)^2.$$

That is,

$$(\tfrac{1}{2}xy)^2 + (\tfrac{1}{2}xz)^2 + (\tfrac{1}{2}yz)^2 \stackrel{?}{=} (\tfrac{1}{2}\sqrt{x^2y^2+x^2z^2+y^2z^2})^2$$

Which, each expression being equivalent to

$$\frac{x^2y^2+x^2z^2+y^2z^2}{4},$$

indeed is so. That is,

$$(\text{Area } \triangle AOB)^2 + (\text{Area } \triangle AOC)^2 + (\text{Area } \triangle BOC)^2$$
$$= (\text{Area } \triangle ABC)^2. \quad \text{Q.E.D.}$$

Corollary: If the general equation of a plane in space is

$$Ax + By + Cz + D = 0$$

where neither A, B, nor C is equal to zero, derive a general expression for the area of the triangle formed by the three traces of the plane.

Derivation: From the above demonstration, the area of the triangle under consideration is given by

$$\tfrac{1}{2}\sqrt{x^2y^2+x^2z^2+y^2z^2},$$

where x, y, and z are the x-, y-, and z-intercepts, respectively.

Now, an intercept is found by giving a value of zero to two variables (of x, y, and z) and then, using D, solving for the remaining variable.

Accordingly, the x-intercept:

$$Ax + By + Cz + D = 0$$
$$Ax + 0y + 0z = -D$$
$$Ax = -D$$
$$x\text{-int} = -\frac{D}{A}.$$

In the same manner, y-int $= -\frac{D}{B}$, and z-int $= -\frac{D}{C}$.

Substituting these values in the first area relation yields the general equation

$$\text{Area} = \tfrac{1}{2}\sqrt{x^2y^2+x^2z^2+y^2z^2}$$
$$= \tfrac{1}{2}\sqrt{(-\tfrac{D}{A})^2.(-\tfrac{D}{B})^2 + (-\tfrac{D}{A})^2.(-\tfrac{D}{C})^2 + (-\tfrac{D}{B})^2.(-\tfrac{D}{C})^2}$$

or, because each value is squared:

$$\text{Area} = \tfrac{1}{2}\sqrt{\frac{D^4}{(AB)^2}+\frac{D^4}{(AC)^2}+\frac{D^4}{(BC)^2}}.$$

MAKING OF MATHEMATICAL MODELS from Solid Geometry has been a source of fascination and challenge over the centuries--in fact, since Greek times. The activity lends itself to Math Club, Math Fair, and other out-of-class individual or small-group investigation. These young ladies, Summer School students of the Editor's, built up elaborate models from individual, cut-out faces, with remarkable skill.

Viewing FAA at Work

A UNIQUE CHALLENGE

"Keep Separated" Joking Admonition ...

Tower Competence Commands Respect

by Don Allen

An airport control tower, aerospace consultant James Nikkel points out, normally can be counted upon for "just about the best view in town." Here is evidence--wrap-around picture windows, unobstructed vision in every direction! We gaze out, the five of us, past the bustling activity "in miniature" on the Field far below, to a panorama that stretches from Diamond Head and the skyscrapers of Waikiki, to Pearl Harbor, a stone's throw away. We concede that, even here in Honolulu, a city uniquely endowed with world-class "views," James Nikkel, our genial host, is right.

The five of us, three Ohio student members, one teacher-sponsor, one editor, are delighted guests of the Federal Aviation Administration (FAA) at Hickam Air Force Base, adjoining Pearl Harbor. Hickam Tower "controls" nearby Honolulu International Airport, and we've been promised insights into the world of the civilian Air Traffic Controller and into the wider range of activities--and career possibilities--within FAA.

Five of us are to serve as the eyes and ears of 883 Honolulu Convention registrants--and of 30 000 Mathematical Log readers across the nation. (The possibility of having accommodated, within necessarily limited, very efficiently used, Tower space, all who might have elected such a visit, rather made us think of packing teen-agers into a phone booth--certainly out of the question!) We suspect, and soon know, however, that there'll be much that is interesting, even remarkable, for us to report.

Accompanied on this special occasion by Mathematical Log Activities Editor Deborah S. Patonai, their faculty sponsor, and by the Log Editor, the three active members of Saint Vincent-Saint Mary High School's Mu Alpha Theta chapter in Akron are Stephen Malinak (a 1985 graduate, now at Massachusetts Institute of Technology), Sujata "Sue" Patel (a senior), and Kathy Gerst (a junior).

Our host and guide is M. James Nikkel, Development Consultant to the Pacific Aerospace Museum, being constructed at Honolulu International Airport. He has addressed Honolulu Convention on "Opportunities in Aerospace," and his encyclopedic knowledge of aircraft and of airport procedures--of everything we witness in the air and on the ground--adds immeasurably to the educational impact of the excursion.

The FAA has a number of distinct functions, reflected in rather diverse career opportunities and personnel needs, we learn at the outset. FAA responsibilities include the control of air traffic, the certification of aircraft and of those who fly them, the operation of high-tech navigational aids, and the writing and enforcement of air safety regulations and air traffic procedures. Founded in 1958, the FAA absorbed the former Civil Aeronautics Administration. Students are impressed, they tell me, by the degree of automation, by the small, highly-skilled team--two men and a woman--that suffices to "control" takeoffs, landings, and on-the-ground movements at this, one of the nation's busier international airports. Our young women record in their notes that they are delighted to find that the FAA is, as might have been expected, an equal-opportunity employer.

Like a schoolroom or a language lab in some respects, but like no schoolroom back in Akron in others, the training area proves to be a windowless enclave of radar scopes and computer hardware where Air Traffic Controllers master diverse skills. Simulation is the key, with anything and everything that might happen in the air or on the ground appearing in computer-generated "scenarios" on training screens. No video arcade has for us created such realism, such a multisensory illusion that "you are there!"

Simulation aside, other screens--electronic-age portholes, we'd say!--"track" real targets (each an aircraft) over a bullseye of concentric rings. Keeping the dots apart on the screen might be the training room metaphor for what air traffic control is essentially about. Fascinatingly, the flip of a switch brings plane type, registration number, or airline and flight, to view ... or causes the scale to change, or superimposes an outline of Oahu.

The cool competence impresses!

One participant is equally struck by the redundancy--the back-up--built into the equipment.

A further fascination: the giant reels of slowly turning sound tape, inexorably taking down, preserving, on multiple tracks, every word communicated, on every frequency, between Tower and ground or plane.

On the Field below, students have been intrigued by the exotic mix of the military and the civilian freely visible at Hickam Base. A heavy transport, coming in low ... "Very possibly Okinawa," suggests our host. World War II fighter planes, displayed as historic relics. Bullet marks from the Pearl Harbor attack. A jumbo jet, taxiing above our heads ... where the coral runway, im-

(See "Unique Challenge," page 4)

DAZZLING HAWAIIAN SUNSHINE casts Akron, OH visitors and FAA spokesman in silhouette during a Convention-related Mathematical Log "you were there" excursion to historic Hickam Air Force Base--and to the Tower controlling air traffic at Honolulu International Airport. Visitors were struck by the level of automation, the few people necessary for the task. Other student reactions to FAA-related activities are shared in the accompanying article. Don Allen photo.

UNIQUE CHALLENGE

...FROM PAGE THREE

probably, passes directly above our vehicle's access road.

There is much, on reflection, that is improbable.

Honolulu International Airport, a jet-age crossroads, can be at its bustling busiest (we learn in conversation) in the small hours of the morning ... when trans-Pacific flights touch down to refuel. During refueling, all must leave the aircraft. A 2 a.m. Airport shopping spree!

The Tower itself, set apart at Hickam Field? The ultimate video arcade, with a wrap-around view! ... but for the stern realization, "This all is for real."

Our Tower briefing concluded, questions asked and well answered, that view fully savored, our host calls to the men and woman monitoring those glowing multiple screens ... with that calculated cheeriness that reflects, but also erodes, stress--our thanks, and:

"Keep separated!"

We are half way down the steep metal stairway before it fully strikes us what that traditional admonition really means.

"AN EERIE FEELING ... you could almost visualize the planes coming over the horizon," recalls Ohio sponsor Debbie Patonai (far right) of this Hickam Air Force Base setting. Amid swaying palms and tropical sunlight, pockmarks on walls remain grim reminders of an anything-but-peaceful Sunday forty years ago. On this Log-related visit to Pearl Harbor facilities are the Editor, students Kathryn Gerst, Stephen Malinak, and Sujata Patel, and their faculty sponsor--from Saint Vincent-Saint Mary High School, Akron. The photo is by aerospace consultant James Nikkel, who arranged and conducted the unique tour.

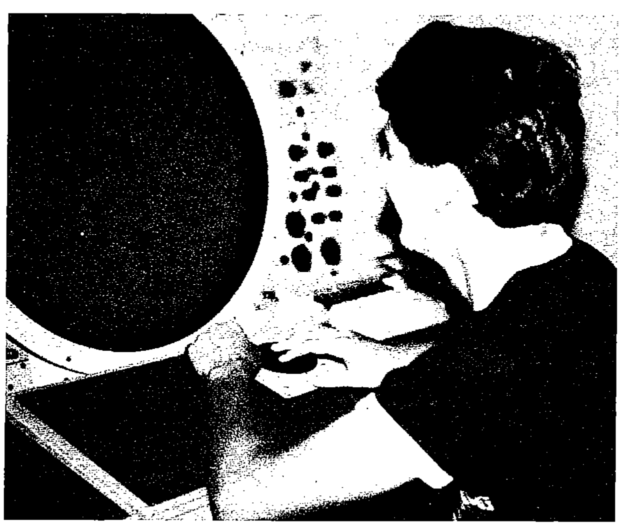
diaLogue

with Log Editor Don Allen

"DiaLogue" is what we like to think The Mathematical Log, National Conventions, and Mu Alpha Theta at its all-important grass-roots level, are all about ... exchange of ideas, insights, perspectives--about mathematics, problem solving, learning, careers, and life itself! Our "diaLogue" column is to facilitate and nurture this kind of exchange--year-round. Contributions are invited!

For those who greet us with a cheery, "Give us another "challenge," we share a further "representation" number, 5 7 3 8. Try for 1 - 100! See rules in April's "diaLogue."

* * *



JET-AGE "SCENARIOS" appear on this training screen, challenging and honing the skills of would-be Air Traffic Controllers. Mu Alpha Theta member Stephen Malinak, 1985 Akron high school graduate, visited Hickam Air Force Base Control Tower while attending Honolulu Convention, then proceeded to Fall studies at MIT. Deborah Patonai photo.

Gary Shaffer, Mu Alpha Theta sponsor at Allegany Community College (Cumberland, MD), continues his Log-inspired experimentation with Lewis Carroll "word linkages" (to Carroll, "Doublets"--see February Log, p. 2), and reports the "circle-squaring" feat of all time! Consider:

C I R C L E
E C L A I R
C L E A R S
C L A U S E
S Q U E A L
S Q U A R E

This "five step," we suspect optimal, solution Gary credits to his sister, Lisa Shank, a middle school mathematics teacher. Note that the solution (above) permits permutation of letters, as needed, plus one letter change at every step. Lewis Carroll might or might not have approved of permutation (see December Log, p. 6), but we think it makes for a better game.

Now, TESSERACT to (say) HYPERBOLA ... in how few steps?

* * *

So,
1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, ..., what, logically, might come next?

31, "of course," then 18, 39, 20, 42, 32, ..., or so we decided in a moment of insight which followed some months with no apparent breakthrough.

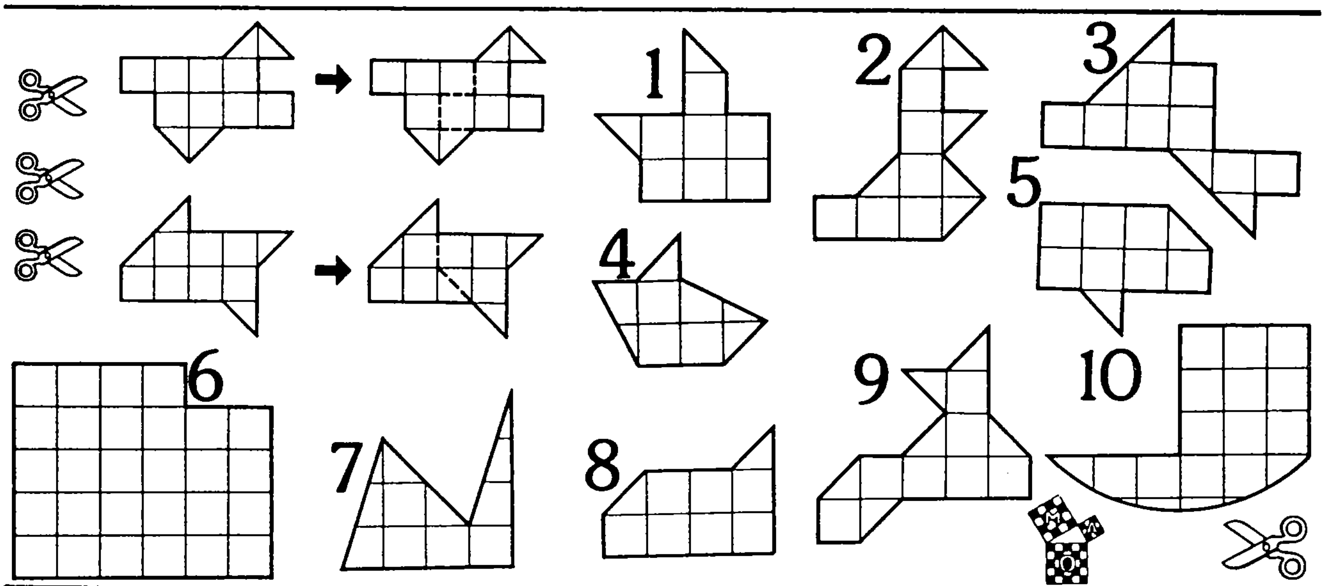
The n^{th} term of the sequence is, we now realize, the sum of the divisors of n ... that and nothing more!

Posed years ago by School Science and Mathematics reader Alna Zame of Coral Gables, FL, the question was revived by this Editor as THETA-21 in Carol McGill's "Logmaster's Choice" (April 1985 Mathematical Log).

No one reported a solution, though the clue, useful by hindsight, is that the function is multiplicative, $u_{ab} = u_a \cdot u_b$ iff a and b are relatively prime. (u_n here denoting the n^{th} term of the sequence. Note that $u_n = n + 1$ iff n is prime.)

Turn to this topic--multiplicative functions--in an introductory reference on Theory of Numbers and you'll find much to consider and to discuss.

(See "diaLogue," page 5)



SCISSORS CHALLENGE from Overseas! Divide each figure into congruent halves, invites Pythagoras, the Dutch-language school math journal with which your Log maintains "exchange" relations, and the challenge is a good one! Two solutions are shown. The remaining ten figures, enlarged and cut from cardboard, might provide a start for an interesting "discovery" session on perceiving congruence. A logical follow-up: inventing new figures to "halve."

diaLogue

... FROM PAGE FOUR

22	25	17	30	19	27	32
8	11	3	16	5	13	18
17	20	12	25	14	22	27
12	15	7	20	9	17	22
9	12	4	17	6	14	19
5	8	0	13	2	10	15
20	23	15	28	17	25	30

The above array of non-negative integers (some repeated) is not a "magic square" in the usual sense of that term (that rows, columns, perhaps diagonals sum to a "magic constant"). It will permit a "magic" feat, nonetheless!

The "trick" should be infallible, and here's how it goes. You first pick any one of the 49 numbers. You circle the chosen number, then cross out the remaining numbers in the row and column in which your number occurred. Right? Now choose one of the remaining numbers, circle it, and cross out the remaining row and column entries. Repeat the process. Continue until seven numbers have been chosen (circled), and all others have been crossed out. Add the numbers that you've circled. The sum will always be the same, in our example 111.

How does the trick work? When you think you know, try to arrange your own array to add to a preselected total. This pattern we developed for our enrichment students, who seemed to enjoy its trickery--but there's good mathematics underlying many such a "fun and games" activity.

Let us know if you've a favorite that you'd care to share.

* * *

"Like virtue, a good puzzle is its own reward," says Problematical Recreations[4], a delightful compilation of "rewarding challenges" put out by Litton Industries some years ago. Here's our favorite from one of these out-of-print pocket collections, recently sent us by a reader:

"A drawer contains an odd number of plain brown socks and an even number of plain black socks. What is the least number of brown and black socks such that the probability of obtaining two brown socks is ½ when two socks are chosen at random from the complete collection?"

* * *

Remember 12 = S[igns] of the Z[odiac], 6 = L[egs] on an I[nsect], and (our favorite) 200 = D[ollars] for P[assing] G[o] in M[onopoly]? Our "Number Associations" chapter activity, shared with us (as "making the rounds") by Mu Alpha Theta sponsor Gary Shaffer (April Log, p. 4), has been traced to its source by Logmaster Todd Belton, Baton Rouge, LA, who very thoughtfully photocopied his voluminous file on this popular activity.

It all began with "Pencilwise" editor Will Shortz' "Equation Analysis Test" in Games magazine, May/June 1981. Much of the same material appeared in Omni of April 1982, with acknowledgement to Games in the July Omni. Two new batches of puzzlers subsequently appeared in Games, incorporating rich gleanings from reader contributions: see "New Equation Analysis Test," May/June 1982, and "In the Last Analysis," November 1982. Which augments the literature of the subject with such gems as:

1 = D at a T
3 = S Y O at the O B G
20 = Y that R V W S
4 + 20 = B B in a P
300 = a P G in B

and our current favorite:

>1 = W to S a C

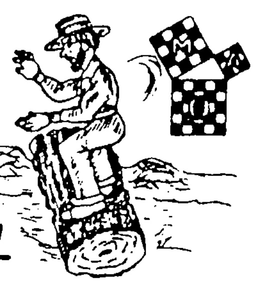
Answers? Well, hunting old Games, Omnis, Scientific Americans, and National Geographics is a pursuit worthy of the best of us! Should all else fail, Todd would be delighted to hear from you at 11622 Pamela, Baton Rouge, LA 70815.

MU ALPHA THETA is people, ideas, and the interaction of people and ideas in a mathematical context. Here, the space geometry of Hawaiian scaffolding frames four Florida stalwarts, math educators Betty and Donovan Lichtenberg (she's National President) and five-year members Phillip AyoungChee and Alan Axelrod, 1986 graduates of North Miami Beach Senior High School. Don Allen photo.

PROBLEM CORNER

LOGMASTER'S CHOICE

with CAROL McGILL



Carol McGill invites solutions, solution notes, additional problems for solution. Look for her at Miama Convention, or write her at 4405 Rue Des Fleurs, Orange, TX 77630.

Carol McGill observes:

Reading the contribution of Wesley Darbro and John Daly in the April Mathematical Log ("'Twin Prime' Study Yields Insights," p. 5) leads me to share with "Problem Corner" readers the concept of "prime triples." Prime triples are usually taken to be a set of primes fitting one of the following patterns:

n, n + 2, n + 4, as in 3, 5, 7;
n, n + 2, n + 6, as in 5, 7, 11;
n, n + 4, n + 6, as in 7, 11, 13; or 13, 17, 19.

I would like to pose questions for the reader--both trivial and not so trivial--based on these patterns.

THETA-39
Three "Prime Triple" Questions

1. There can be but one illustration of the form of prime triple, n, n + 2, n + 4; namely, 3, 5, 7. Why is this so?

2. List as many as you can of prime triples of the form n, n + 2, n + 6. Note that the first two sets (5, 7, 11, and 11, 13, 17) add to primes if you sum their terms: 5 + 7 + 11 = 23, and 11 + 13 + 17 = 41. Will this be true of all prime triples of this form? What conjectures might be made about prime triples of this form, especially about their distribution?

3. List as many as you can of prime triples of the form n, n + 4, n + 6. What conjectures might be made about prime triples of this form, especially about their distribution?

* * *

"Some general comments about primes, before closing, may be in order," Carol writes. "Most readers have met the Sieve of Eratosthenes, developed more than 2000 years ago and used to predict primes. The American mathematician D. H. Lehmer published in 1914 a tabulation of primes to ten million. Even 'before computers,' such listing had been carried to 100 million! Today, with computer aid, American mathematician Bryant Tuckerman has shown

$$2^{19\,937} - 1$$

to be prime--a number of 6002 digits."

* * *

The Editor, who gets first try at "Problem Corner" and who (space allowing) is permitted the last word, finds "prime triples" a new and interesting concept. What about such "prime quadruples," conspicuous in base ten, as 101, 103, 107, 109, and 191, 193, 197, 199? And what of the analogues of prime triples and prime quadruples suggested in our "prize number" digression, April 1984 Mathematical Log, p. 4?

* * *

The natural numbers include many interesting subsets, with new categories and their properties making for innovative fun from time to time. Who knows of "wondrous numbers"? Father Stanley Bezuszka, Boston College, introduced them to a General Session audience at Mu Alpha Theta Honolulu Convention last July. Take any [natural] number. It, necessarily, is odd or even. If it is odd, triple it, then add 1. If it is even, divide by 2. Continue the process. Wondrous numbers are those numbers from which you reach 1. Thus, 31 reaches 1, in 106 steps, and 27 reaches 1, in 111 steps, making 31 and 27 wondrous numbers, Father Bezuszka notes. Can you find a formula for the number of steps? Can you confirm that all numbers are wondrous numbers? Much was left to think about for that Honolulu capacity audience.

Our Summer School students had, by coincidence, been studying properties of the same numbers and number operations--and asked for something a bit different. That was when we came up with the "trichotomy" procedure we state below:

THETA-40
A "Trichotomy" Procedure

Choose any counting number. Either it is a multiple of three, one more than a multiple of three, or one less than a multiple of three. If it is a multiple of three, divide by three. If it is one more than a multiple of three, multiply by two then add one. If it is one less than a multiple of three, multiply by two then subtract one. Verify that, repeating the procedure as often as necessary, you reach one. Determine the number of numbers from which one can be reached in exactly 10 steps. ...exactly n steps.

For readers wishing further insight into the nature of the suggested procedure, an "opening up" of the question will be found in our current Tall Timbers.

* * *

Charles D. Gallant, our colleague at Nova Scotia's St. Francis Xavier University, recently shared with us an elementary but innovative problem that he felt might be of math club interest. We conclude with it.

THETA-41
Positive, Composite, and Relatively Prime

Write seven positive integers which are composite, relatively prime, and less than 361.

Now, try for eight!