CONVENTION, BACK-TO-SCHOOL ISSUE

The Mathematical Log

Volume 28, Number 3 October 1984

# STRAIGHTEDGE-COMPASS CONSTRUCTION APPROXIMATES ANGLE TRISECTION

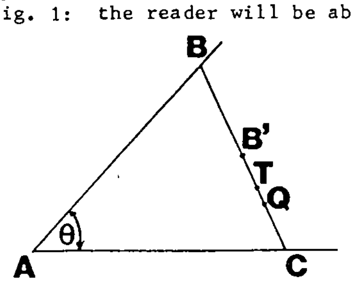
BY MARC A. P. BERNSTEIN

The ancient problem of trisecting an angle using only a compass and straightedge is revisited. A simple construction is described which approximately trisects any angle less than 180° with a maximum error of 1/3°. This approximation can be improved by decomposing the original angle to a multiple of 5 5/8° plus an angle smaller than 5 5/8°. Since 5 5/8° can be trisected exactly, the problem is reduced to trisecting an angle smaller than 5 5/8°. The present construction can trisect such angles with a maximum error of $3.4 \times 10^{-5}$ degree.

This construction is compared with the construction which uses the original angle as the apex of an isosceles triangle, trisects the base, and connects the apex with the trisection points. The present construction is shown to induce errors a factor of 10 smaller than the latter construction for angles less than 20°. The ratio of errors improves for larger angles to a factor of between 15 and 100.

Finally the present construction is compared with the procedure of quadrasecting the angle, adding this to the quadrasection of the quadrasection, etc. Although this last technique converges to a perfect angle trisection, it requires infinite iterations. To set a perspective for comparison, the number of iterations resulting in an error comparable to that incurred in the present construction is noted. It is shown that for angles less than 5 5/8°, the errors are comparable to those incurred with about 9 iterations of the quadrasection. Over the whole range, the errors are about the same as those incurred with about 4 iterations of the quadrasection.

Construction. Given an arbitrary angle, θ, at vertex A construct the line BC so that AB = AC and angle BAC = θ. Mark the points Q, T, and B' which quadrasect, trisect, and bisect BC respectively--i.e., QC = BC/4, TC = BC/3, and B'C = BC/2 (Fig. 1: the reader will be able to, and




should, perform the rest of the construction as an exercise--Ed.). With A as center and radius AB draw the arc BC. With B as center and radius BC draw semicircle BC. At point Q draw a perpendicular to BC intersecting semicircle BC at Q'. Determine the point, X, along the extension of line BC which is the apex of the isosceles triangle with TQ' as its base. A simple procedure to determine point X uses the result B'X = 3(B'C) - B'T. Using X as center and radius XT swing an arc intersecting arc BC (the arc of the circle centered at A) at point Y. Draw AY. Angle YAC is the approximation to the trisection of θ.

Assessment of Accuracy. In order to assess the accuracy of the present construction a simple construction was made. The original angle, θ, at vertex A was made the apex of an isosceles triangle, ABC. The base, BC, was trisected at point T (BT = BC/3) and line AT was constructed. The angle BAT was used as the approximation to θ/3.

A further perspective was gained by calculating the result of a finite number of iterations of successive quadrasections of the original angle. The original angle was bisected twice (first iteration). One of the quarters was then bisected twice and the result added to the original quarter (second iteration). The process was repeated a finite number of times.

The result of these three techniques is summarized in Table 1. Column 1 contains the original angle, θ. Column 2 lists the error in degrees incurred in approximating the trisection of θ by the present method--i.e. column 2 gives the resultant angle from the present construction - θ/3. Column 3 presents the error resulting from a finite number of iterations of successive quadrasections. The number of iterations, adjusted according to θ, used is indicated in column 5.

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## GREAT MU ALPHA THETA 'MATH-PACK' CHALLENGE

--PAGE THREE.

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Table 1 reveals several interesting conclusions. The present construction is most accurate at the extremeties (0° and 180°). The greatest error encountered is about 1/3° when θ = 140°. The errors encountered in both other constructions are minimum at θ = 0° and increase monotonically with θ. The present construction occasions errors at least a factor of 10 lower than those encountered in the base trisection scheme. In order to reduce the errors in the successive quadrasection construction to approximate those of the present construction, 9 iterations are required for 1° < θ < 10°, 8 iterations are required for 10° < θ < 20°, 6 iterations for 20° < θ < 30 , and 4 for 40° < θ < 180°. The present construction always yields a positive error (approximate trisection exceeds θ/3) while the other techniques yield negative errors.

Since the very small errors occur near θ = 0° it is worthwhile exploring how to reduce the general problem to the trisection of small angles. Clearly one can construct a 90° angle and also a 30° angle. By bisecting each of these angles 4 times one can construct a 5 5/8° (5.625°)

(Continued on page 6)

The Mathematical Log

ISSN 0025-5580

The Mathematical Log, with Mathematical Tall Timbers, a supplement for Chapters, is the official publication of the National High School and Junior College Mathematics Club, Mu Alpha Theta (M A Th). Mu Alpha Theta is sponsored by the Mathematical Association of America and the National Council of Teachers of Mathematics. The Log is published four times each year, in February, April, October, and December. Editorial deadline is two months prior to publication. Correspondence should be addressed to Mu Alpha Theta, 601 Elm Ave., Rm. 423, University of Oklahoma, Norman OK 73019.

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# Convention Special 1984

PLAN AHEAD FOR 1985

Mu·Alpha·Theta
Honolulu·Hawaii·1985

ELECTED AND APPOINTED OFFICERS, 1983-84



# dia Log ue

with Log Editor Don Allen

WE ENJOY AMERICA at "grassroots" level, and have found Mu Alpha Theta convention time an ideal time and opportunity for meandering the byways—and meeting the "real people"—of, to date, 41 states. Hence the Greyhound, No. 4824, en route north and east from Oklahoma, "somewhere in Indiana." We especially enjoy Mu Alpha Theta people, and value interaction with them—in person at Convention and through the year by Log and mail. Hence the "portraiture"! We do encourage Editor-spotting, iff it can lead to the dialogue that will permit us—and our editorial team—to do a more effective job.


Bus 4824—now there's a number! And, like so many a real-world number-label, 4824 has potential for some long-distance-travel fun. Take those four digits—4, 8, 2, 4. Write them, in any order (let's agree). Use them as digits, addends, factors, exponents, etc. Connect them by signs for addition, subtraction, multiplication, division—used any finite number of times. Allow brackets, decimals, repeating decimals ($.\overline{2} = 2/9$), the surd for square root. Starting with

$1 = (8 \times 2)/(4 \times 4)$

how many consecutive positive integers can you represent? Clearly,

$2 = 84/42$, or $48/24$
$3 = 28/4 - 4$
$4 = (24 - 8)/4$
$5 = (8 \times 2 + 4)/4$
. . .

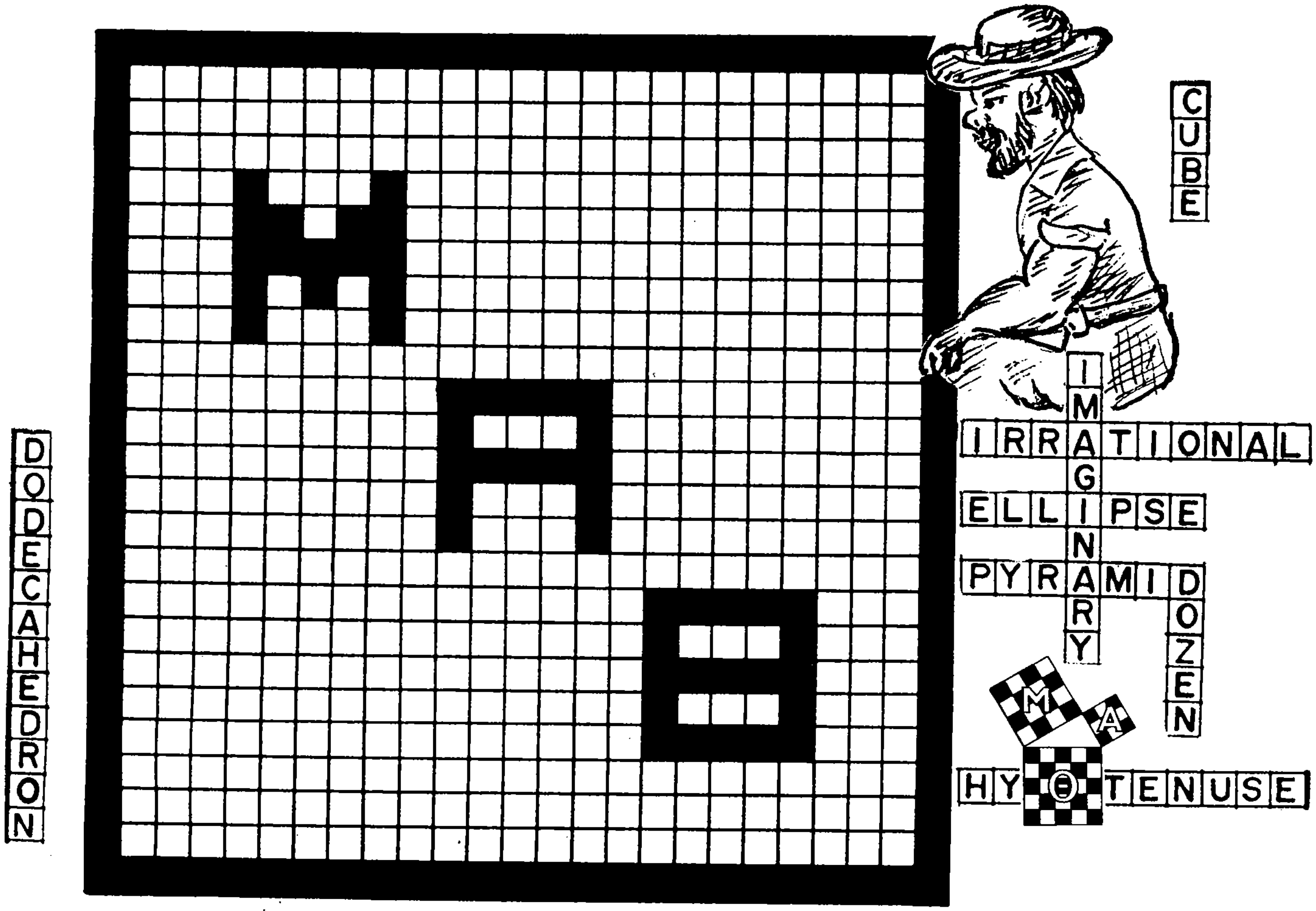
Be alert to "priority of operations"—multiplication "takes precedence over" addition. Use of brackets should help you to "say what you mean."

We've initiated such "representation" competitions in previous Logs (3692, Winter 1981; 5741, December 1982), and they've proven popular. 4824 is such a competition—with a difference. Even integers cause little difficulty. Odd integers can be another matter. But each of the first 100 positive integers can be represented using a 4, an 8, a 2, and a 4, we now know. We had all the even ones while still on the bus. The odd ones took somewhat longer! Let's keep in touch.

# HAWAII IN '85

With plans moving ahead rapidly for Mu Alpha Theta's 1985 national convention—July 30 - August 5 in Hawaii—interested schools and sponsors may wish to contact the convention chairman, Jeanne F. Nelson, The Kamehameha Schools, Kapalama Heights, Honolulu, HI 96817. Business and residence telephones are 808/842-8511 and 808/395-5468. Watch your Log and Tall Timbers, too, for further details.

# GREAT MU ALPHA THETA 'MATH-PACK' CHALLENGE




## BY DON ALLEN

Mathematics-related "crossword challenge" competitions, contests in which mathematical terms are to be so "packed" into specified "grids" as to achieve maximum point scores, have consistently led as the most popular Mathematical Log reader activities in recent years. Such "word packing" competitions have featured mathematicians' names (Winter 1981), geometric terms (February 1982), measurement units (February 1983), and numeration concepts (February 1984), in each instance the words to be "packed" being stipulated in contest rules. Convention/back-to-school time being challenge and competition time in Mu Alpha Theta publications, Log "word packing" tradition is continued--and extended--in a new reader contest which (innovatively) leaves the player relatively free to seek and choose "optimal" terms for the "packing" competition.

A distinctive "Mu Alpha Theta" grid, being introduced for this activity, offers 469 accessible square cells. Filled (colored) cells--the letters Mu, Alpha, and Theta --reduce symmetry, and add to the challenge. "Math words" (names, terms, etc.), of four or more letters, are to be entered, horizontally (left to right) or vertically (top to bottom), one letter to a cell. Words may "cross," as in a crossword puzzle--in fact, all words must be so connected (there can be no "isolated" words or clusters of words. However, in every other sense, individual words must be separated (an empty--or colored--cell must "come between" words, horizontally or vertically). The letters of words will score points. The object of the activity is to achieve the maximum possible score.

Selected words, let's agree, must be mathematics related--nouns, verbs, and adjectives referring to concepts, structures, figures, operations. Names of mathematicians are acceptable, surnames in their usual English spelling. No hyphenated words, plural or possessive forms, or Roman numerals. A suitable word may be "played" more than once, but (to add challenge) not more than three times.

"MATH PACK" LETTER VALUES

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | 5 | H | 6 | N | 5 | T | 4 |
| B | 10 | I | 6 | O | 5 | U | 8 |
| C | 8 | J | 12 | P | 10 | V | 10 |
| D | 8 | K | 12 | Q | 12 | W | 10 |
| E | 3 | L | 8 | R | 6 | X | 12 |
| F | 10 | M | 8 | S | 6 | Y | 10 |
| G | 10 | | | | | Z | 12 |

Letters of words entered on the grid will score points on a sliding scale--higher point values being associated with the less-common English letters. Thus,

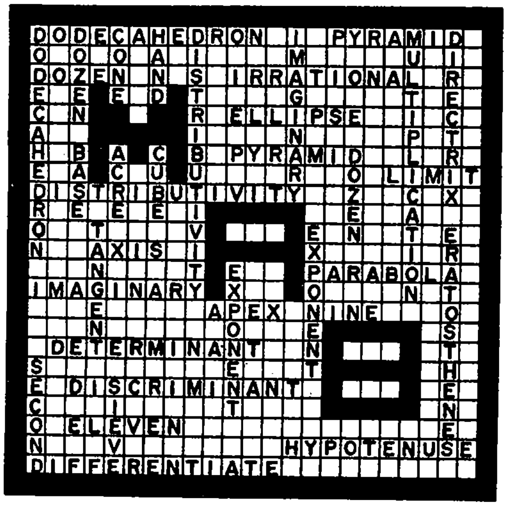
(Continued on page 4)

# WZSUUHOW HU WKKJ, QYKRHOW HU XSBBSY. – QKTPC

## 'MATH-PACK' CHALLENGE

FROM PAGE THREE

nine "high-frequency" letters, E T A O N I R S H, rate 3 to 6 points (see table); five "medium frequency" letters, D L U C M, 8 points; seven "low-frequency" letters, P F Y W G B V, 10 points; and five "rare letters" (relatively speaking), J K Q X Z, earn 12 points each. Any letter common to two words (that is, representing their grid intersection) is, in effect, counted twice.



"WORD-PACKING" TECHNIQUES are effectively demonstrated in the author's trial solution (above), as is selection of suitable mathematics-related words, but the 1733-point total score (check this!) is one that should be fairly easy to beat. Thirty-six word entries, including seven intentional duplications, sum to the indicated total, and serve to show word intersection (SIEVE and ELEVEN, TANGENT and AXIS, representatively) and separation (APEX and NINE, DOZEN and BASE, with DETERMINANT and SECOND "touching" only at a corner, which is permissible). Note that each DISTRIBUTIVITY scores 8 + 6 + 6 + 4 + 6 + 6 + 10 + 8 + 4 + 6 + 10 + 6 + 4 + 10, or 94 . . . 188 for the two. MULTIPLICATION, similarly, scores 91; NINE a mere 19. The Great Math-Pack Challenge: to produce the highest possible score . . . exactly how high we'd not hazard to guess. This particular solution, parenthetically, serves to fill 243 of 469 accessible cells, a tie-breaking consideration.

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Accordingly, QUOTIENT scores 12 + 8 + 5 + 4 + 6 + 3 + 5 + 4, or 47; PARALLEL scores 10 + 5 + 6 + 5 + 2x8 + 3 + 8, or 53; ARCHIMEDES scores 5 + 6 + 8 + 6 + 6 + 8 + 3 + 8 + 3 + 6, or 59.

The highest total point score wins. It's as simple as that. In the event of a tie, the solution leaving the fewer unoccupied cells will be judged winner.

Open to individuals and groups within Mu Alpha Theta, our "Great Math-Pack Competition" will extend through three "phases." Phase I, when the premium likely will be on speed, will be the New Orleans "on-site phase," with a modest prize for the best solution sub-

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-mitted by Convention closing session. Phase II, the "cross-country phase," will recognize superior achievement at school and chapter level, through 31 October postmark deadline. Phase III, our "championship phase," will call upon all and sundry to outdo Phase I and II accomplishments, as reported in December Log.

## NEW JERSEY LOGMASTER TOP GUESSER

"Choose and send in on a postcard a Fibonacci number (apart from the initial 1's) which will be the least such number to be submitted by exactly one person!"

Such was the Mathematical Log "Fibonacci Challenge" (December Log). Logmaster Jeff Slovin, Eatontown, NJ, won easily with a finite number of separate postcard submissions (actually, five) from the infinitude of Fibonacci possibilities. Jeff's winning Fibonacci number was 3.

Jeff, to play safe, also guessed 2, but so did Kathryn Kobe, Bishop Ward High School chapter, Kansas City, KS. Jeff also guessed 8, 21, and 34. Omitted by Jeff, 5 was submitted by Shari Berkenblit, Mu Alpha Theta vice-president at North Valley Stream, NY. The highest prediction also came from Bishop Ward chapter: Dennis Bindolski guessed the 18th Fibonacci number, 2584.

Fibonacci numbers derive from the infinite sequence 1, 1, 2, 3, 5, 8, 13, ..., where each number after the second is the sum of the two preceding numbers.

The Mathematical Log extends thanks to competition participants. As a neighborly gesture, results of this "outguessing" fun activity are being communicated to our good friends in the Fibonacci Association. H.D.A.

## 4TH DIMENSION FASCINATES 1983 KALIN AWARD WINNER

"I may never design a hyper-drive ship or formulate a Grand Unification Theory, but Mu Alpha Theta has helped me to begin to use my abilities for the pursuit and sharing of knowledge." So states Roger Kirpes, Dubuque, IA, who gave a paper on the Fourth Dimension at St. Louis convention. Roger, former Math Club president at Wahlert High School, Dubuque, and former Mu Alpha Theta national student president, was Kalin Award winner at Norman, OK, national convention.

"Mu Alpha Theta has given me the chance to pursue my math interests and share what I learn with others. My interest in the Fourth Dimension was born as a result of a talk I heard, and I began to read on my own," Roger reports. He wrote a paper, and gave a presentation on the subject 18 months later.

Roger's plans were to major in mathematics and physics with an astronomy emphasis at Iowa State University. Eventually he hopes to teach or do research in theoretical physics or astronomy.

Roger's high school resume reveals the wide range of interests and commitments characteristic of successful Kalin candidates. Representative are math and physics competitions, biology olympics, French club, band and orchestra, ten years in Scouting, conservation activities, parish council and other religious activities. He also has had six poems published, and served as poetry editor of his school literary magazine.

Discussing his school mathematics experiences, Roger has noted:

"As an underclassman, I spent much time studying math and participated in many tournaments. My love of math became an insatiable hunger as I discovered that there was so much more to math, all of which I wanted to learn. I spent much of my free time in independent study, and as a junior I took senior math. As a senior I created a one-person calculus class—me. But my knowledge is still miniscule and my hunger only whetted. I have a need to learn, and an even greater need to share with others what I have learned."

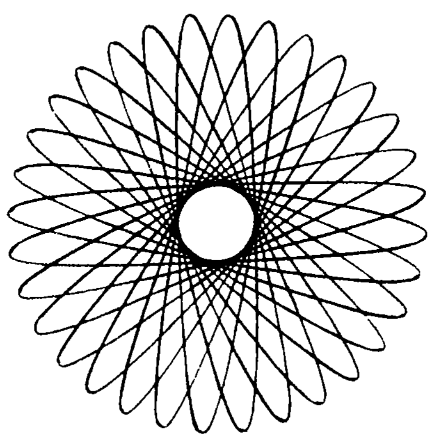
When last we saw Roger he was on Norman campus—his head under the hood of an indisposed but Iowa-bound vehicle. We'll wager he made it home ... and will welcome his further reflections in years ahead.

## 'Design Toy' Investigated

# KEY TO REPETITIVE PATTERNS FOUND IN RATIOS, MULTIPLES

By Jeffrey Scott Crews

Kenner's Spirograph is a well known "educational design toy" consisting of toothed plastic wheels, rings, and other pieces which are revolved around each other with a pen to generate a great diversity of continuous designs. The principle governing these designs is simple. To explain it, the simplest case, that of a wheel revolving within a ring, is used. The pieces have definite numbers of teeth: the wheel has (W) teeth, and the ring (R) teeth. Each design, as well, has a definite number of "points." "Points" or "bends" are sharp turns in the design close to the ring, where the pen makes its closest approaches to the ring.



EXAMPLE 1: A 45 tooth wheel rotating in a 96 tooth ring.

L.C.M. Method:
45 = 3 x 3 x 5
96 = 2 x 2 x 2 x 2 x 2 x 3
L.C.M. (45, 96) = 2 x 2 x 2 x 2 x 2 x 3 x 3 x 5
= 1440
1440 ÷ 96 = 15, circumscribings of the ring.
1440 ÷ 45 = 32, wheel rotations and points.

Ratio Method:
45:96 ... divide out common factor of 3
15:32 ... wheel side of ratio is the number of circumscribings of the ring, ring side is the number of rotations of the wheel and number of points.
15 circumscribings
32 points

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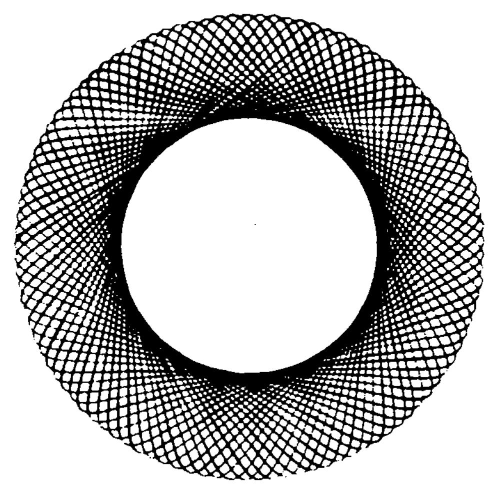
A design is finished when the pen returns to its starting point and the pattern begins to repeat. When this happens, a tooth on the wheel also returns to the tooth on the ring at which it began. Example: Tooth A on the wheel is at tooth A' on the ring when the drawing begins. The drawing is complete when tooth A returns to tooth A'. However, for a tooth on the wheel to return to the same tooth on the ring at which it began, two things must be true: The ring must be "gone around," "circled," or "circumscribed" by the wheel (r) number of times, where (r) is an integer, and the wheel must rotate (w) times, where (w) is an integer also.

When the wheel rotates, every tooth meshes (gets touched by another) once for every rotation. Thus with (w) rotations, there are (W x w) meshings for a wheel with (W) teeth. Likewise, when a ring is circled, each tooth on the ring is touched once for every circling, so that with (r) circlings there are (R x r) meshings for a ring with (R) teeth.

However, since every meshing requires a tooth from the wheel and one from the ring, the number of meshings for each is the same. Thus, W x w = R x r, and since all the variables are integers, W x w (or R x r) is the least common multiple of W and R. (If it is less, then the drawing will not be complete, and if it is more, then the drawing will already have begun to repeat.) Now, since there are (W x w) meshings, and the wheel has (W) teeth, any particular tooth, such as the tooth closest to the pen, meshes (W x w) ÷ W, or w times. These close approaches to the pen, remember, are points or bends on the design, so that there are (w) points. By the same method, there will be (R x r) ÷ R, or r circlings or circumscribings of the ring by the wheel. These results can be summarized as follows:



EXAMPLE 2: A 32 tooth wheel rotating in a 105 tooth ring.

L.C.M. Method:
32 = 2 x 2 x 2 x 2 x 2
105 = 3 x 5 x 7
L.C.M. (32, 105) = 2 x 2 x 2 x 2 x 2 x 3 x 5 x 7
= 3360
3360 ÷ 105 = 32, circumscribings of the ring.
3360 ÷ 32 = 105, wheel rotations and points.

Ratio Method:
32:105 ... relatively prime, there are no common factors
32 circumscribings
105 wheel rotations and points

This always happens when there are no common factors: a point for every tooth on the stationary piece.

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The number of points in a design is the least common multiple of the numbers of teeth on the two pieces, divided by the number of teeth on the piece that is moving, and the number of circlings of the stationary piece is the L.C.M. divided by the number of teeth on the stationary piece.

(Continued on page 6)