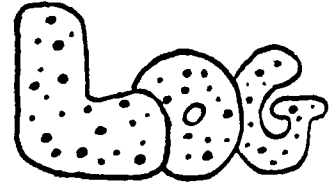
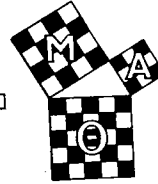
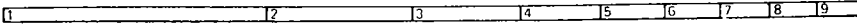


THE MATHEMATICAL

VOLUME 28, NUMBER 2

APRIL 1984



Steiner-Lehmus Theorem

Angle Bisector Equality Considered In Varied, Challenging Proofs

by

Ali R. Amir-Moéz and Gregory A. Fredricks

Even though synthetic geometry is not fashionable these days, we still get requests for a proof of the so-called Steiner-Lehmus theorem. Thus it would seem proper to have a few proofs available.

Theorem: If the angle bisectors of two angles of a triangle are equal, the triangle is isosceles.

An indirect proof: In order to be brief, we shall often refer to a diagram (Fig. 1). Let ABC be the triangle whose angle bisectors

are equal, and we denote the measure of the congruent angles by μ . Letting $\lambda = \angle DLC$ and $\delta = \angle LDC$ we have angles as marked in Fig. 1. Comparing triangles EOB and DOC , we see that

$$\beta + \lambda + \mu + \sigma = \sigma + \mu + \delta + \gamma$$

and hence

$$\beta - \gamma = \delta - \lambda. \quad (*)$$

If $\beta > \gamma$, then $DC > CL$, and hence $\lambda > \delta$ in triangle LDC . This contradicts (*). If $\beta < \gamma$, then we get $\lambda < \delta$ which again contradicts (*).

A direct proof: Construct the triangle ECF congruent to the triangle BDC (Fig. 2). Thus $EF = BC$. Note that the figure is marked for

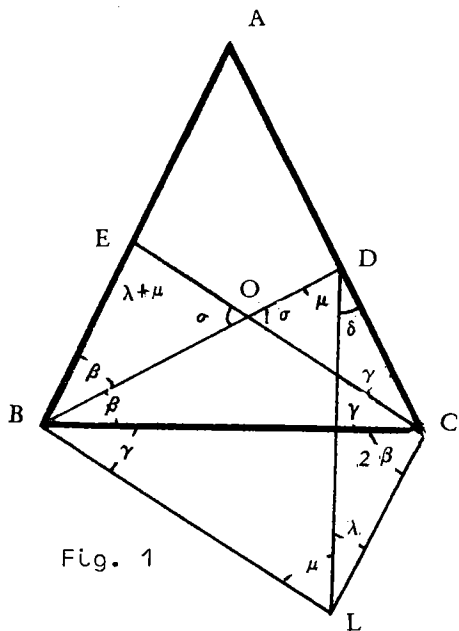


Fig. 1

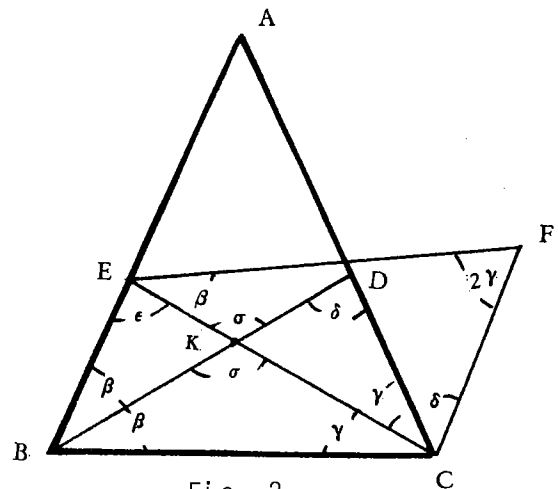


Fig. 2

of B and C are equal, i.e., $BD = CE$. Note that the half angles are denoted by β and γ . In fact other angles also are marked.

Construct the parallelogram $ECLB$. It follows from the hypothesis that triangle BLD is

further reference. One observes that $\sigma = \beta + \epsilon = \angle FEB = \delta + \gamma = \angle FCB$. It now follows that $EFCB$ is a parallelogram (why?). Therefore $2\beta = 2\gamma$.

Indeed, the ever present problem of order of half lines creates a difficulty. One has somehow to show that whenever two angle bisectors are equal, corresponding angles have to be acute (why?).

An algebraic proof: One may compute the

(Continued on page 2)



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Steiner-Lehmus Theorem

FROM PAGE 1

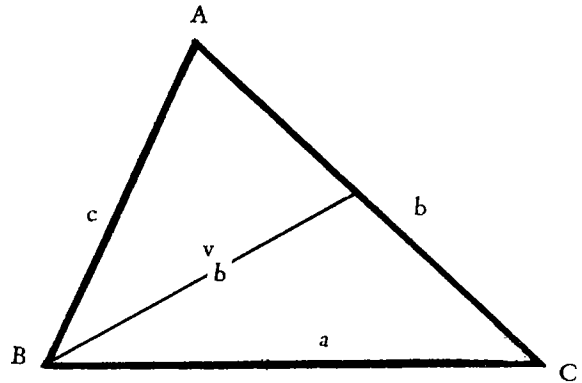


Fig. 3

lengths of the angle bisectors. Let a, b, c , be the lengths of the sides of a triangle, and v_b, v_c be the angle bisectors at B and C (Fig. 3). Then

$$v_b^2 = ca[(c + a)^2 - b^2]/(b + c)^2,$$

$$v_c^2 = ab[(a + b)^2 - c^2]/(a + b)^2.$$

If one sets $v_b = v_c$, having done some factoring one obtains

$$(b - c) \{ (bc/(c + a))^2 (a + b)^2 [a^2 + 2a(b + c) + b^2 + bc + c^2] + 1 \} = 0$$

Showing that the expression inside the braces is positive, one obtains $b = c$ (see School Science and Mathematics, January 1960).

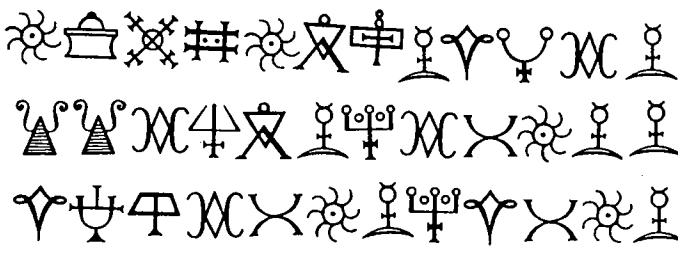
For a vector proof, also instructive, refer to your 1968 Logs.

Ali R. Amir-Moéz and Gregory A. Fredricks are with the Department of Mathematics, Texas Tech University, Lubbock, TX 79409.

Logging ... in April

A TEO MIEN A TEO!

'Hyhl1t1' (one of our favorite literary figures!) said it, and said it well ... and his philosophical reflection on self and limitations (p. 1) ably sets the tone of a varied, we trust interesting April Log. (Yes, with so few words and symbols, the quote nonetheless can be "solved.") Have fun, too, with the eloquent claim to mathematical prowess, enciphered in Alchemy! And do let us know how you did. H.D.A.





Changing the Sieve

'Prize' Number Properties Instructive to Explore

By Don Allen

The prime number sequence (2, 3, 5, 7, 11, ...), known since antiquity, can be fun to develop and remarkably instructive to explore. The prime numbers, for one thing, permit a surprising diversity of elementary conjectures--many of them more readily made than proven or disproven, history has shown. The prime numbers are not the only such sequence possible, however. (Ulam developed and explored the "lucky numbers," so-called.) The "sifting" procedure commonly used to yield an efficient listing of prime numbers (the Sieve of Eratosthenes) can readily be varied to yield other, prime-like sequences. You may wish to try this, very possibly producing your own sequence. For a start, however, let me offer "prize numbers" (2, 3, 7, 9, 13, ...), developed to give my own students experience in this kind of exploration.

Eratosthenes' classic "sieve," as now used, begins with the naturals (1, 2, 3, 4, 5, ...). 1, the first natural, is set aside as the unit, and unique. 2, accordingly, is the first prime number. After 2, every 2nd natural is struck out. 3, which remains, is the next prime number. After 3, every 3rd natural, whether already "eliminated" or not, is struck out. 5 is the next prime number. After 5, every 5th natural is struck out. This process eliminates multiples. Left are primes--numbers whose only divisors, clearly, are 1 (the unit) and themselves.

Exactly how might the classic sieve be varied? My students have found it remarkably dif-

STANDARD CURRICULUM

UTTFXWK YWJ QUXVCXWK, MG RMHULT, VM ZTKXW QXVC; YWJ VCTW VCT JXGGTUTWV ZUYWRCTL MG YUXVCDTVXR--YDZXVXMW, JXLVUYRVXMW, HKFXGX- RYVXMW, YWJ JTUXLXMW. --VCT DMRP VHUVFT.

ficult to come up with a useful and instructive result. However, a sequence of multiples (as struck out by Eratosthenes) might be viewed as a special case of an arithmetical progression--and, for me, that provided the key. Prize numbers result from use of the following modified sieve.

Start (as before) with the naturals. Set aside 1, the first natural, as the unit, neither prize nor (let's say) a loser. Take 2 as the first prize number. After 2, strike out each 2nd natural number. 3 remains, and is the next prize number. After 3, count 2 (the previous prize), and strike out 5 and each 3rd subsequent number; that is, the subsequent numbers of the form $3k + 2$. 7 remains, and is the next prize number. After 7, count 3, and strike out 10, and subsequent numbers of the form $7k + 3$. 9 is the next prize number. Strike out 16, and subsequent numbers of the form $9k + 7$. Generalizing, if z_m is the m th prize number, strike out subsequent numbers of the form $z_m(k) + z_{m-1}$.

The sequence of remaining naturals gives

the prize numbers: 2, 3, 7, 9, 13, 15, 19, 21, 27, 33, 37, 39, 49, 51, 55, 57, 63, 67, 69, 81, 85, 99,

Note that, just as there were "twin primes" (pairs of odd primes differing by 2, as 17, 19; 59, 61), so there are "twin prizes" (67, 69; 175, 177). Many other such analogies should be fairly easy to spot.

You might wish to attempt investigations along some of the following lines:

Primes, after 2 and 3, all necessarily are one less or one more than a multiple of six: that is, such primes may be written $6k - 1$ or $6k + 1$. What are corresponding forms for prize numbers? Why? Debouville once conjectured that for all k , one or both of $6k - 1$, $6k + 1$ would be prime. Was he right? Could there be a corresponding conjecture for prize numbers?

Euler is credited with discovering the remarkable quadratic expression $n^2 - n + 41$, which yields prime values for $n = 1, 2, 3, \dots$ (for how long?). Is there a corresponding expression "rich" in prize values?

There are numbers which are both prime and prize: 2, 3, 7, 13, 19, 37, 67, 109, 127, After 2, 3, all such numbers are of the form $6k + 1$. Is there something special about the sequence of k -values yielding such numbers?

The Greeks were well aware of "the infinitude of primes." Euclid showed, by indirect proof, that there could be no "last" prime number. What of the prize numbers? If we suppose there are only k prize numbers, that z_k is the last prize number, what can we say about the number represented by

$$(2k_1)(3k_2 + 2)(7k_3 + 3)(9k_4 + 7) \dots$$

$$(z_{k_k} + z_{k-1}) + 1? \text{ (the } k_i \text{ are constants)}$$

The Greeks classified numbers as deficient, abundant, or perfect, according as whether the sum of the proper divisors was less than, greater than, or equal to the number. ($28 = 14 + 7 + 4 + 2 + 1$ equalled the sum of the proper divisors, and so was "perfect.") Now, 5 (say) is

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eliminated as non-prize, being of the form $3k + 2$: we might term 3 the "eliminator." 88 has four eliminators: 2, 9, 15, 49; and 178 has five. Might the number of eliminators or the sum of the eliminators prove interesting to investigate?

Primes can be colored on checkerboards or graphed in other instructive ways. Could prize numbers be similarly treated to advantage?

The possibilities would seem limited only by the imagination or by the ability to "stay with" a self-set task. Do let us know your "prize" discoveries, or properties of other sequences you develop through a modified sifting process.

